Corner Polyhedra and Maximal Lattice-free Convex Sets : A Geometric Approach to Cutting Plane Theory

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Abstract

Corner Polyhedra were introduced by Gomory in the early 60s and were studied by Gomory and Johnson. The importance of the corner polyhedron is underscored by the fact that almost all “generic” cutting planes, both in the theoretical literature as well as ones used in practice, are valid for the corner polyhedron. So the corner polyhedron can be viewed as a unifying structure from which many of the known cutting planes can be derived. Moreover, the corner polyhedron has recently attracted a lot of attention as a potential source of new kinds of cutting planes.

It has also been observed that valid inequalities for the corner polyhedron have an intimate connection with maximal lattice-free convex sets in $\mathbb{R}^m$. This connection provides a geometric way of understanding and analyzing these valid inequalities. Such an approach often yields new insights into properties of existing cutting planes, as well as provides a novel way of analyzing new families of cutting planes.

The goal of this thesis is to explore this interplay of maximal lattice-free convex bodies and cutting plane theory. We also hope to obtain new insights into the structure of the corner polyhedron with the help of this approach.
1 Introduction

Mixed Integer Linear Programs (MILPs) are a general framework for modeling an extremely large number of problems in technological and scientific applications. Mathematically, they can be viewed as a generalization of the problem of solving systems of linear Diophantine equations, where we also allow inequalities in the system. Since in general MILPs fall in the NP-complete class of problems, we have no hope of obtaining polynomial algorithms for solving them in general. Even so, algorithms have been developed which have other theoretical guarantees and also have been engineered to work remarkably well in practice. A very successful approach in this direction has been the cutting-plane method. Originally proposed by Gomory [29], this algorithm can be proved to terminate in finite time for pure integer problems (where all variables are required to be integer-valued). Moreover, the finite termination of this method can also be proved for mixed 0-1 problems as a consequence of Balas’s sequential convexification result [4]. See also [6, 7]. Moreover, on the practical side, this method has seen a great revival in the 1990s with its effectiveness being truly exploited by clever engineering, robust implementation and increased computing power [8, 20]. Since then all successful commercial codes for solving MILPs (for instance, Xpress and Cplex) have relied heavily on the branch-and-cut framework and cutting planes are an extremely effective method for solving large MILPs today. In fact, MILPs which would have required years of computation time 20 years ago can now be solved in seconds today (see Bixby et al.[10] for a survey of these developments). In spite of this, there are several examples of MILPs (in particular combinatorial auctions and sports scheduling) which are beyond the scope of current state-of-the-art software. Advances in cutting plane and integer programming theory would be a main factor in achieving such speed-ups. This is one of the major motivations behind recent investigations in this field.

Moreover, cutting plane theory makes for very elegant mathematics. Consider a system of inequalities \(Ax \leq b\) where \(x\) is a vector of variables \(x_1, \ldots, x_n\), \(A\) is an \(m \times n\) matrix and \(b\) is a \(m \times 1\) column vector. So it is a system of \(m\) inequalities in \(n\) variables. We wish to find integral solutions to the system, i.e. \(x \in \mathbb{Z}^m\) such that \(Ax \leq b\). The term integer hull is used to refer to the convex hull of all such \(x\). The basic idea behind cutting plane theory is to approximate the integer hull as well as possible by introducing additional linear inequalities into the system. These linear inequalities are such that every solution of the original system satisfy the new inequalities also. Such linear inequalities are classically called valid inequalities or cutting planes for the solution set. We will define these notions formally in Section 1.4.1.
The set \{x \in \mathbb{R}^m : Ax \leq b\}, where we have relaxed the integrality constraints on the variables, will be referred to as the \textit{linear relaxation} of the original problem. Such a set is called a \textit{polyhedron} in \mathbb{R}^m. The interesting cutting planes are ones which cut off some portions of the linear relaxation. Figure 1(a) illustrates this geometric idea in two dimensions. The black lines define the original inequalities of the system and hence form the linear relaxation. The blue lines are examples of cutting planes or valid inequalities. The red lines show the integer hull, i.e. the convex hull of the integer solutions. Mixed Integer Linear Programs (MILPs) add the further generality that only a subset of the variables \(x_1, \ldots, x_n\) are required to be integer-valued. The notions of integer hull, linear relaxation and cutting planes generalize in a straightforward manner to such problems. See Figure 1(b) - the light red lines show the entire solution set, the dark red lines show the integer hull.

The main goal of this thesis is to provide new insights into cutting plane theory using the lens of corner polyhedra and lattice-free convex sets.

\section{Corner Polyhedra}

Corner polyhedra were introduced by Gomory [30] as a generalization of his original cutting plane method [29] and this structure was studied extensively by Gomory and Johnson [31, 35]. We introduce this concept and the notation in this section.

Consider a general MILP in the following standard form. Let \(A\) be a \(m \times n\) matrix, \(b\), be \(m \times 1\) column vectors. We are interested in \(x \in \mathbb{R}^n\) such that

\[
Ax = b, \quad x \geq 0, \quad x_j \in \mathbb{Z} \text{ for } j \in I \subseteq \{1, \ldots n\}
\]

We assume, without loss of generality, that \(A\) has full row rank. A basis \(B\) is a maximal linearly independent set of columns in \(A\); so \(B\) is an invertible \(m \times m\) matrix. Corresponding to any basis \(B\), we define the corner polyhedron with respect to \(B\), denoted by \textit{corner}(\(B\)) as the set of all \(x \in \mathbb{R}^n\) such that
where $x_B$ refers to the variables corresponding to the columns of $B$ (a.k.a the basic variables) and $x_N$ refers to the remaining variables (a.k.a the non-basic variables). $\text{corner}(B)$ is a relaxation of (1) because the non-negativity constraints on the basic variables have been dropped (a set $P'$ is a relaxation of a set $P$ in $\mathbb{R}^n$ if and only if $P \subseteq P'$). This implies that valid inequalities for $\text{corner}(B)$ are also valid inequalities for (1). The significance of this corner polyhedron lies in the fact that most cutting planes for general MILPs (e.g. GMI cuts, Split cuts, Reduce-and-Split cuts, MIR cuts and so forth) are obtained by solving the linear relaxation using linear programming methods and then using integrality arguments to find an inequality which cuts off this fractional (vertex) solution. Many families of such cutting planes, including the four examples listed above, are valid inequalities for $\text{corner}(B)$ for some appropriate basis $B$. Moreover, for the derivation of these cutting planes, one needs to only consider a relaxation of the corner polyhedra obtained by retaining only one equality constraint. Therefore, having a characterization of polyhedra in $\mathbb{R}$ defined using constraints like (2) would be an automatic generalization of most cutting planes used in practice. In addition, such polyhedra have valid inequalities (facets) which cannot be obtained by applying integrality arguments to a single constraint from the system [2]. So the analysis of corner polyhedra is likely to provide us with new families of cutting planes which may be quite useful for solving general MILPs. We now represent (2) in a slightly more general notation which we will use for the rest of this document.

\[
\begin{align*}
x &= f + \sum_{i=1}^{k} r^i s_i \\
s_i &\geq 0 \quad \text{for all } i \\
x &\in \mathbb{Z}^m, s_i \in \mathbb{Z}, i \in I \text{ for some subset } I \subseteq \{1, \ldots, k\}
\end{align*}
\] (3)

In the above, $f \in \mathbb{R}^m$ corresponds to $B^{-1}b$ in (2) and $r^i$s are rays in $\mathbb{R}^m$ which correspond to the non-basic columns ($-B^{-1}N_i$) in (2) ($N_i$ denotes the $i$-th column of the matrix $N$). $x$ corresponds to $x_B$ and $s$ corresponds to $x_N$. We make a few comments about the system 3 below.

**Remark 1.** Note that in (3) we have imposed the condition that $x$ be integer-valued, whereas in (2), all the variables in $x_B$ need not have integrality constraints. But note that the constraints corresponding to these variables can be safely dropped from the system without changing the solution set. Hence, in our reformulation (3), we impose integrality constraints on all the variables in $x$.

**Remark 2.** Moreover, in (2) we had integrality constraints on some subset of the non-basic variables $x_N$. Therefore, we impose integrality constraints only on a subset of the $s_i$ variables (this is denoted by $I$ in (3)). So in general the set of solutions to (3) is a mixed-integer set - some variables taking real values while others are restricted to be integer-valued.

**Remark 3.** Note further that we only need to record the values of the $s_i$ variables from the solution set for (3) - the values of the $x$ variables are automatically determined. This is a simple, but important point because in the rest of the document we will be looking at variants and generalizations of the system (3) and we will only concentrate on the space of the $s_i$ variables.
1.2 Generalizations of the basic mixed integer set

1.2.1 Pure Integer sets and the Group Problem

Consider the problem (3) where \( I = \{1, \ldots, k\} \), that is all the \( s_i \) variables have integrality constraints on them. Such a problem will be referred to as a finite group problem. We denote the fractional part of a ray \( r \in \mathbb{R}^m \) by \( \lfloor r \rfloor = r \mod 1 \), where the modulo is taken component-wise. As noted in Remark 3, we can concentrate on the \( s \)-space. So we denote by \( G_f(r_1, \ldots, r_k) \) the set of all points \( s \) such that the following holds.

\[
 f + \sum_{i=1}^{k} r^i s_i \in \mathbb{Z}^m, \quad s_i \in \mathbb{Z}, \quad s_i \geq 0 \quad \text{for all} \ i
\]

(4)

which is the same as the set of all points \( s \) such that

\[
 f + \sum_{i=1}^{k} \lfloor r^i \rfloor s_i \in \mathbb{Z}^m, \quad s_i \in \mathbb{Z}, \quad s_i \geq 0 \quad \text{for all} \ i
\]

where the difference from (4) is that we only consider the fractional part of each ray \( r^i \). The term group comes from the following interpretation of \( G_f(r_1, \ldots, r_k) \). Consider the Abelian group generated by \( \{\lfloor r_1 \rfloor, \ldots, \lfloor r_k \rfloor\} \) with addition modulo 1. Then \( G_f(r_1, \ldots, r_k) \) can be interpreted as those \( s \) such that the particular element corresponding to \( s \) in this Abelian group is the same as \( -f \).

Gomory and Johnson [31, 33] suggested the following generalization. We put in all elements \( r \in [0,1]^m \) in this group problem and a variable \( s_r \) corresponding to each group element \( r \in [0,1]^m \). We are interested in all points \( s \) with finite support (\( s_r > 0 \) for only finitely many \( r \)) such that

\[
 f + \sum_{r \in [0,1]^m} r s_r \in \mathbb{Z}^m, \quad s_r \in \mathbb{Z}, \quad s_r \geq 0 \quad \forall r \in [0,1]^m
\]

We denote by \( G_f \) the set of all such \( s \). This problem is referred to as the infinite group problem. This set lives in an infinite dimensional linear space. The advantage of studying such a set is that any finite group problem \( G_f(r_1, \ldots, r_k) \) can be obtained by restricting \( G_f \) to the subspace where \( s_r = 0 \) if \( r \notin \{\lfloor r_1 \rfloor, \ldots, \lfloor r_k \rfloor\} \) (See Theorem 1 in [33]). So it is worthwhile to study the set \( G_f \) also because valid inequalities for \( G_f \) are going to give valid inequalities for any finite group problem when restricted to that appropriate subspace, and hence for any pure integer programming problem. This is emphasized in [33].

1.2.2 Relaxing integrality of the non-basic variables

Consider the original mixed integer set defined by the system (3). If we relax the integrality constraints on \( s_i, i \in I \) then we get a further relaxation of our problem and its solution set will be denoted by \( R_f(r_1, \ldots, r_k) \). As pointed out in Remark 3 we can concentrate on the space of \( s \) variables. Formally, \( R_f(r_1, \ldots, r_k) \) denotes the set of all points \( s \) such that

\[
 f + \sum_{i=1}^{k} r^j s_i \in \mathbb{Z}^m, \quad s_i \geq 0 \quad \text{for all} \ i
\]

(5)
Such a problem was analyzed by Andersen et al. in [1] with \( m = 2 \). They showed connections between valid inequalities for the convex hull of \( R_f(r^1, \ldots, r^k) \) and lattice-free convex sets. This connection is further explored in [2] for general \( m \). Since \( R_f(r^1, \ldots, r^k) \) is a superset of the points which satisfy (3), valid inequalities for \( R_f(r^1, \ldots, r^k) \) will be valid for (3).

Analogous to \( G_f \) we can define the set \( R_f \) by introducing a variable \( s_r \) for every \( r \in \mathbb{R}^m \) and considering all \( s_r \) with finite support (i.e. \( s_r > 0 \) for finitely many rays \( r \in \mathbb{R}^m \)) which satisfy the following constraints.

\[
f + \sum_{r \in \mathbb{R}^m} r s_r \in \mathbb{Z}^m, \quad s_r \geq 0 \quad \forall r \in \mathbb{R}^m
\]

This problem will be called the semi-infinite relaxation. As with \( G_f \) and \( G_f(r^1, \ldots, r^k) \) we have that \( R_f(r^1, \ldots, r^k) \) can be obtained by restricting \( R_f \) to the subspace \( s_r = 0 \) for \( r \notin \{r^1, \ldots, r^k\} \) (See Fact 2 in [12]). This again implies that valid inequalities for \( R_f \) give valid inequalities for \( R_f(r^1, \ldots, r^k) \) when restricted to that appropriate subspace.

The set \( R_f \) restricted to the space of all rational rays (i.e. \( \mathbb{Q}^m \)) was characterized by Borozan and Cornuejols in [16]. The correspondence between minimal valid inequalities and maximal lattice-free sets is exhibited in that paper and as a consequence, it is shown that minimal valid inequalities for \( R_f \) (restricted to the rational rays) always correspond to piecewise linear functions. This will be explained further in the later sections of this document.

1.2.3 The mixed integer group problem

We can combine the generalizations from the two previous subsections to define a master problem which contains all the previous problems as its faces. Define a variable \( u_r \) for each \( r \in [0, 1]^m \) and \( w_r \) for each \( r \in \mathbb{R}^m \). Then we can define \( M_f \) to be the set of all \( u_r \) and \( w_r \) such that

\[
f + \sum_{r \in [0,1]^m} r u_r + \sum_{r \in \mathbb{R}^m} r w_r \in \mathbb{Z}^m, \quad u_r \in \mathbb{Z}, u_r \geq 0 \quad \forall r \in [0,1]^m, \quad w_r \geq 0 \quad \forall r \in \mathbb{R}^m
\]

This problem is referred to as the mixed integer group problem and was first studied by Johnson [35]. Dey and Wolsey in [25] exploit the connection with lattice-free sets for studying minimal and extremal inequalities for this problem.

The important thing to note is that the infinite group problem \( G_f \) as well as the semi-infinite relaxation \( R_f \) are both faces of \( M_f \), i.e. they can be obtained from \( M_f \) by restricting it to a proper subspace. For example, \( R_f \) is obtained by intersecting \( M_f \) with the subspace where \( u_r = 0 \) for all \( r \in [0, 1]^m \). Consequently \( G_f(r^1, \ldots, r^k) \) and \( R_f(r^1, \ldots, r^k) \) are faces of \( M_f \). Moreover, note that the original corner polyhedra with integer AND continuous variables (as defined by a system like (3)) is also contained as a face of \( M_f \) in a similar manner. To state this less formally, \( M_f \) contains all possible corner polyhedra and its variants described above in it.
1.3 Maximal Lattice-free Convex sets

Early work on maximal lattice-free convex sets dates back to Minkowski. A wealth of results were obtained following the work of Lenstra [37]. See Lovász [38] for a survey. We outline some basic results in this topic that will be needed in this document.

We will be concerned with convex sets in linear subspaces of $\mathbb{R}^n$. Given $X \subset \mathbb{R}^n$, we denote by $\langle X \rangle$ the linear space generated by the vectors in $X$. The underlying field is $\mathbb{R}$.

**Definition 4.** An additive group $\Lambda$ of $\mathbb{R}^n$ is said to be finitely generated if there exist vectors $a_1, \ldots, a_m \in \mathbb{R}^n$ such that $\Lambda = \{\lambda_1 a_1 + \ldots + \lambda_m a_m | \lambda_1, \ldots, \lambda_m \in \mathbb{Z}\}$.

If a finitely generated additive group $\Lambda$ of $\mathbb{R}^n$ can be generated by linearly independent vectors $a_1, \ldots, a_m$, then $\Lambda$ is called a lattice of the linear space $\langle a_1, \ldots, a_m \rangle$. The set of vectors $a_1, \ldots, a_m$ is called a basis of the lattice $\Lambda$.

**Definition 5.** Let $\Lambda$ be a lattice of a linear space $V$ of $\mathbb{R}^n$. Given a linear subspace $L$ of $V$, we say that $L$ is a lattice-subspace of $V$ if there exists a basis of $L$ contained in $\Lambda$.

Given $y \in \mathbb{R}^n$ and $\varepsilon > 0$, we will denote by $B_\varepsilon(y)$ the open ball centered at $y$ of radius $\varepsilon$. Given an affine space $W$ of $\mathbb{R}^n$ and a set $S \subseteq W$, we denote by $\text{int}_W(S)$ the interior of $S$ with respect to the topology induced on $W$ by $\mathbb{R}^n$, namely $\text{int}_W(S)$ is the set of points $x \in S$ such that $B_\varepsilon(x) \cap W \subset S$ for some $\varepsilon > 0$. We use the notation $\text{aff}(S)$ to denote the smallest affine space containing $S$. We denote by $\text{relint}(S)$ the relative interior of $S$, that is $\text{relint}(S) = \text{int}_{\text{aff}(S)}(S)$. $\dim(S)$ denotes the dimension of $\text{aff}(S)$.

**Definition 6.** Let $\Lambda$ be a lattice of a linear space $V$ of $\mathbb{R}^n$, and let $W$ be a linear space of $\mathbb{R}^n$ containing $V$. A set $S \subset \mathbb{R}^n$ is said to be a maximal $\Lambda$-free convex set of $W$ if $S \subset W$, $S$ is convex, $\Lambda \cap \text{int}_W(S) = \emptyset$, and $S$ is an inclusion-wise maximal set with these properties.

**Theorem 7.** Let $\Lambda$ be a lattice of a linear space $V$ of $\mathbb{R}^n$, and let $W$ be a linear space of $\mathbb{R}^n$ containing $V$. A set $S \subset \mathbb{R}^n$ is a maximal $\Lambda$-free convex set of $W$ if and only if one of the following holds:

(i) $S$ is a polyhedron in $W$, $\dim(S) = \dim(W)$, $S \cap V$ is a maximal $\Lambda$-free convex set $P$ of $V$, and for every facet $F$ of $S$, $F \cap V$ is a facet of $P$;

(ii) $S$ is an affine hyperplane of $W$ of the form $S = v + L$ where $v \in S$ and $L \cap V$ is a hyperplane of $V$ that is not a lattice subspace of $V$;

(iii) $S$ is a half-space of $W$ that contains $V$ on its boundary.

A characterization of maximal $\Lambda$-free convex sets of $V$, needed in (i) of the previous theorem, is given by the following.

**Theorem 8.** Let $\Lambda$ be a lattice of a linear space $V$ of $\mathbb{R}^n$. A set $S \subset \mathbb{R}^n$ is a maximal $\Lambda$-free convex set of $V$ if and only if one of the following holds:

(i) $\dim(S) = \dim(V)$, $S = P + L$ where $L$ is a lattice-subspace of $V$, $P$ is a polytope in the orthogonal complement of $L$, $\Lambda \cap \text{relint}(P) = \emptyset$, and there is a point of $\Lambda$ in the relative interior of each facet of $S$.
(ii) \( \dim(S) < \dim(V) \), \( S \) is an affine hyperplane of \( V \) of the form \( S = v + L \) where \( v \in S \) and \( L \) is not a lattice-subspace of \( V \).

Theorem 7 is new. Theorem 8 is due to Lovász ([38] Proposition 3.1). Lovász only gives a sketch of the proof and it is not clear how case (ii) in the above theorem arises in his sketch or in the statement of his proposition. Therefore, we prove both theorems in [12] for the sake of completeness.

1.4 Mathematical Preliminaries and Notation

We formally define some fundamental notions related to convex sets and functions in this section.

1.4.1 Valid Inequalities for a convex set

For any set \( S \) in a linear space \( V \) (over the field \( \mathbb{R} \)) we denote the convex hull of \( S \) by \( \text{conv}(S) \). In a normed linear space \( (V, \| \cdot \|) \), the closure of a set \( S \subseteq V \) will be denoted by \( \overline{S} \). Given a convex set \( S \) in a vector space \( V \), a valid inequality for \( S \) is defined by a linear functional \( \Psi \) on \( V \) and \( \alpha \in \mathbb{R} \) such that \( \Psi(s) \geq \alpha \) for all \( s \in S \). Let \( B \) be a basis for \( V \) and \( s_b, b \in B \) be the coordinates of any point \( s \in S \) in the basis \( B \). Any linear functional \( \Psi \) can be represented as \( \sum_{b \in B} \psi(b)s_b \) where \( \psi \) is a mapping from \( B \) to \( \mathbb{R} \) [18]. So a valid inequality for \( S \) becomes \( \sum_{b \in B} \psi(b)s_b \geq \alpha \). Such a mapping \( \psi \) will be called a valid function for \( S \). For any valid linear inequality if \( \alpha > 0 \), we can divide out by \( \alpha \) and get an equivalent inequality of the form \( \sum_{r \in B} \psi(r)s_r \geq 1 \). Similarly, if \( \alpha < 0 \), we can rewrite the inequality as \( \sum_{r \in B} \psi(r)s_r \geq -1 \). So we have three types of valid linear inequalities, namely with RHS -1, 0 and 1.

In the rest of this document, we will be often be concerned with convex sets \( S \) which only contain points with non-negative coordinates in the basis \( B \). For such sets we can define a hierarchy of valid inequalities, with an increasing notion of “strength”. In the remainder of this section, we will assume \( S \) is such a convex set.

A minimal valid inequality for \( S \) is defined by some valid function \( \psi \) such that there exists no other valid function \( \psi' \) for \( S \) with \( \psi'(b) \leq \psi(b) \) for all \( b \in B \) and \( \psi'(b) < \psi(b) \) for some \( b \in B \). This notion is introduced to formalize the notion of “strong” inequalities. For instance, consider a valid inequality \( \psi \) which is not minimal, one can find a valid inequality \( \psi' \) such that \( \sum_{b \in B} \psi'(b)s_b \geq \alpha \) cuts off all the points cut off by \( \sum_{b \in B} \psi(b)s_b \geq \alpha \) and cuts off at least one more point. So \( \psi' \) dominates \( \psi \).

An extreme valid inequality for \( S \) is defined by some valid function \( \psi \) for \( S \) such that there do not exist valid functions \( \psi_1 \) and \( \psi_2 \) for \( S \) such that \( \psi(b) \geq \frac{1}{2} \psi_1(b) + \frac{1}{2} \psi_2(b) \) for all \( b \in B \).

It is known [31] that extreme valid inequalities must be minimal and hence extreme valid inequalities are a subset of minimal valid inequalities, which are themselves a subset of valid inequalities.

Given a valid function \( \psi \) for \( S \), denote by \( E(\psi) \) the points \( s \in S \) with \( \sum_{b \in B} \psi(b)s_b = \alpha \), i.e. the points from \( S \) which satisfy the inequality at equality. A facet is defined by a valid function \( \psi \) for \( S \) such that for any valid function \( \psi' \) for \( S \), \( E(\psi) \subseteq E(\psi') \) implies \( \psi = \psi' \).
1.4.2 Definitions

Consider a linear space $V$ over some ordered field $K$ (like $\mathbb{Q}$ or $\mathbb{R}$). Consider a function $\psi : V \to K$. Such a function is called sub-additive if $\psi(a + b) \leq \psi(a) + \psi(b)$ for all $a, b \in V$. $\psi$ is said to be positively homogeneous if $\psi(cr) = c\psi(r)$ for all $c \in K$ and $r \in V$. When the field $K$ is the field $\mathbb{R}$, $\psi$ is said to be convex if $\psi(\lambda a + (1 - \lambda)b) \leq \lambda\psi(a) + (1 - \lambda)\psi(b)$ for all $a, b \in V$ and $\lambda \in [0, 1]$.

2 Connections between Maximal Lattice-Free Convex Sets and Valid Inequalities for MILPs

We now illustrate the connections between the concepts introduced in the Introduction. As was emphasized in the Introduction, we obtain relaxations and generalizations of our original MILP (1) such that valid inequalities for (1) can be derived from valid inequalities for these relaxations and generalizations. So we concentrate on analyzing the valid inequalities for the following different convex sets - $G_f(r_1, \ldots, r_k)$, $G_f$, $R_f(r_1, \ldots, r_k)$, $R_f$ and $M_f$.

2.1 The sets $R_f$ and $R_f(r_1, \ldots, r_k)$

It was first observed in [1] that valid inequalities for $R_f(r_1, \ldots, r_k)$ in the case of $m = 2$ (i.e. with two equality constraints) correspond to lattice-free convex sets (See Lemma 2 in [1]). Lattice-free convex sets are simply sets which satisfy Definition 6 without the maximality constraint. This observation is a reinterpretation of the classical intersection cuts proposed by Balas [6]. Subsequently, Borozan and Cornuejols [16] established this connection between minimal valid inequalities for $R_f$ and maximal lattice-free convex sets. In particular, they studied the set $R_f$ restricted to the space of rational rays and showed in [16] that the minimal valid inequalities (or equivalently minimal valid functions) are in a one-to-one correspondence with maximal lattice-free convex sets.

We will now formally state the result from Borozan and Cornuejols [16]. This will also be convenient because the notation that we will introduce to describe their result will be used later in this document.

The set $R_f$ lives in the vector space which consists of points $s = (s_r)_{r \in \mathbb{R}^m}$ which have finite support. Let us generalize the concept of $R_f$ for any subset $V$ of $\mathbb{R}^m$ as follows. Consider a point $f \in \mathbb{R}^q \setminus \mathbb{Z}^q$, and a subset $V \subseteq \mathbb{R}^q$ such that $f + \langle V \rangle$ contains an integral point. Let $R_f(V)$ be the set of points $s = (s_r)_{r \in V}$ of $\mathbb{R}^V$ satisfying

\[
\begin{align*}
& f + \sum_{r \in V} rs_r \in \mathbb{Z}^q \\
& s_r \geq 0, \quad r \in V \\
& s \in \mathcal{V}_V
\end{align*}
\]

where $\mathcal{V}_V$ is the set of all $s \in \mathbb{R}^V$ with finite support, i.e. the set $\{ r \in V \mid s_r > 0 \}$ has finite cardinality. In this context, we can interpret valid inequalities defined in Section 1.4.1 as $\sum_{r \in V} \psi(r)s_r \geq \alpha$ for some $\psi : V \to \mathbb{R}$. Also, for simplicity of notation we will often refer to $\sum_{r \in V} \psi(r)s_r$ by $\Psi(s)$ when the set $V$ is clear from the context.
Given a valid inequality \( \sum_{r \in \mathbb{R}} \psi(r)s_r \geq \alpha \), where \( \alpha \in \{-1, 0, 1\} \) we consider the following set

\[
B_\psi = \{ x \in \mathbb{R}^m : x - f \in V, \psi(x - f) \leq \alpha \}.
\]

Borozan and Cornuejols studied \( R_f(\mathbb{Q}^m) \) where \( \mathbb{Q}^m \) is the subset of all vectors with rational components and \( f \in \mathbb{Q}^m \setminus \mathbb{Z}^m \). Moreover, their definition of minimal valid inequality allows \(+\infty\) as a possible value for \( \psi \). But this technicality is not crucial to the understanding of their results.

**Theorem 9.** [16] A minimal valid function \( \psi \) for \( R_f(\mathbb{Q}^m) \) is nonnegative, piecewise linear, positively homogeneous and convex. Furthermore the set \( \overline{B}_\psi \) is a full-dimensional maximal lattice-free convex set containing \( f \). Conversely, for any full-dimensional maximal lattice-free convex set \( B \subseteq \mathbb{R}^m \) containing \( f \), there exists a minimal valid function \( \psi \) for \( R_f(\mathbb{Q}^m) \) such that \( \overline{B}_\psi = B \), and when \( f \) is in the interior of \( B \), this function is unique.

This theorem has been generalized to handle minimal valid inequalities for \( R_f(V) \) when \( V \) is any lattice-subspace of \( \mathbb{R}^m \) in [12]. The same paper has results of a similar nature when \( V \) is any general subspace (not necessarily a lattice-subspace). Theorem 9 and its generalizations from [12] are based on the following characterization of minimal valid functions for \( R_f(V) \).

**Lemma 10.** Let \( \sum_{r \in V} \psi(r)s_r \geq \alpha \) be a minimal valid inequality for \( R_f(V) \). Then the following hold

1. \( \psi(0) = 0 \);
2. \( \psi \) is sub-additive;
3. \( \psi \) is positively homogeneous;
4. \( \psi \) is convex.

The proof of Lemma 10 is identical to one provided by Borozan and Cornuéjols [16] under a rationality assumption.

**2.1.1 Characterization of \( \text{conv}(R_f(V)) \) in terms of lattice-free sets**

The results outlined in the previous section imply a nice characterization of \( \text{conv}(R_f(V)) \) for general subspaces \( V \) (\( \text{conv}(S) \) denotes the convex hull of a set \( S \)). We will need to introduce a little more notation to state these results.

Functions satisfying the conditions of Lemma 10 will be called *convex ray functions*. When a convex ray function \( \psi : V \to \mathbb{R} \) is nonnegative, it is known as a *convex distance function*. See Cassels [22].

Since \( \psi \) is convex, then by standard convex analysis [36] \( \psi \) is continuous in \( V \).

**Remark 11.** A convex ray function is continuous in \( V \).
Since $\psi$ is continuous, $B_\psi$ is closed. Since $\psi(0) = 0$, $f \in \text{int}_V(B_\psi)$ when $\alpha = 1$, $f \in \partial(B_\psi)$ when $\alpha = 0$ and $f \notin B_\psi$ when $\alpha = -1$. Since $\psi$ is convex, $B_\psi$ is convex. Since $\psi$ is a valid function, $B_\psi$ is lattice free. This is exactly like [16] where this correspondence was defined for rational rays. See also the chapter on distance functions in [22]. Moreover, the closed convex sets that we get from this construction have special structure, which we list below.

A body is a closed subset $B \subseteq f + V$ having interior points, i.e. $\text{int}_{f+V}(B) \neq \emptyset$. A star body $B$ with respect to $f$ is a body such that $f \in \text{int}_{f+V}(B)$ and for every $r \in V$ the half-line $f + \lambda r$, $\lambda \geq 0$ either lies entirely in $B$ or there exits $\lambda_r > 0$ such that $f + \lambda r \in B$ for $0 \leq \lambda \leq \lambda_r$ and $f + \lambda r \notin B$ for $\lambda > \lambda_r$. A reverse star body with respect to $f$ is the closure of the complement of a star body with respect to $f$. In other words, for any direction, either the half line in that direction from $f$ is completely outside $B$ or it remains inside $B$ after some point.

Observation 1. A convex body is the intersection of half-spaces bounded by its supporting hyperplanes. A convex reverse star body $B$ with respect to $f$ can be represented as $B = \{x | s \cdot (x - f) \leq -1\}$ for all $s \in S$ for some set $S \subseteq V$. $S$ is the set of vectors $s$ such that $s \cdot (x - f) = -1$ is a supporting hyperplane for $B$.

Observation 2.
1. When $\alpha = 1$, $B_\psi$ is a convex body with $f$ in the interior.
2. When $\alpha = 0$, $B_\psi$ is a closed cone with apex $f$.
3. When $\alpha = -1$, $B_\psi$ is a reverse star body with respect to $f$.

Conversely, consider any lattice-free body $B$ satisfying one of the conditions in Observation 2. Because of its special structure $B$ can be represented as $B = \{x | s \cdot (x - f) \leq \alpha\}$ with $\alpha \in \{-1, 0, 1\}$ and $s \in S$. $S$ is the set of vectors $s$ such that $s \cdot (x - f) = \alpha$ is a supporting hyperplane for $B$ (see Observation 1). For any $r \in V$ we define

$$\psi_B(r) = \sup_{s \in S} s \cdot r$$

It is easy to check that $\psi_B$ satisfies the conditions in Lemma 10 and $\sum_{r \in V} \psi_B(r)s_r \geq \alpha$ is a valid inequality, where $\alpha = \{-1, 0, 1\}$ depending on whether $f$ is outside, on the boundary or in the interior of $B$. Such a definition is very similar to support functions from classical convex analysis [36].

We prove the following theorems in [12].

Theorem 12. If $\sum_{r \in V} \psi(r)s_r \geq \alpha$ is a minimal valid inequality with $\alpha \in \{-1, 1\}$, then $\psi = \psi_B$.

Theorem 12 implies the following theorem as shown in [12]. Let

$$B_V = \{B \subseteq V : B \text{ is a lattice-free convex body of } V \text{ satisfying condition 1 or condition 3 in Observation 2}\}.$$}

In other words, $B_V$ is the set containing all lattice-free convex bodies with $f$ in the interior and all lattice-free reverse star bodies with respect to $f$. 

11
Theorem 13.  
\[
\text{conv}(R_f(V)) = \left\{ s \in \mathcal{V}_V : \sum_{r \in V} \psi_B(r)s_r \geq \alpha \quad B \in \mathcal{B}_V \right\}
\]
where \( \alpha \) is \(-1\) or \(1\) depending on whether \( f \) is inside or outside \( B \).

The closure in the above theorem is defined in terms of a norm which is described in [12]. The technical details will not be of much use in this document. The consequences of these theorems in the special case when \( V \) is a rational subspace are particularly nice because the relevant lattice-free sets are maximal. We collect these results with self-contained proofs in the following subsection.

2.1.2 Rational subspaces

In this section we consider the special case where \( V \) is a lattice-subspace of \( \mathbb{R}^m \) for the standard lattice \( \mathbb{Z}^m \).

Lemma 14. Let \( \psi : V \to \mathbb{R} \) and \( \alpha \in \mathbb{R} \) such that \( \sum_{r \in V} \psi(r)s_r \geq \alpha \) is a nontrivial valid inequality for \( \text{conv}(R_f(V)) \). Then \( \psi(r) \geq 0 \) for every \( r \in V \) and \( \alpha > 0 \).

Proof. Since \( V \) is a lattice-subspace for the standard lattice \( \mathbb{Z}^m \), \( V \cap \mathbb{Q} \) is dense in \( V \). By Remark 11, \( \psi \) is continuous. Thus we only need to show that, for every \( r \in V \cap \mathbb{Q} \), \( \psi(r) \geq 0 \). Suppose \( \psi(\bar{r}) < 0 \) for some \( \bar{r} \in V \cap \mathbb{Q} \). Let \( D \) be an integer such that \( D\bar{r} \) is an integral vector. Given an integral point \( \bar{x} \) in \( f + V \), let \( s^\lambda \in \mathcal{V}_V \) be defined
\[
s^\lambda_r = \begin{cases} 
D\lambda, & \text{if } r = \bar{r}; \\
1, & \text{if } r = \bar{x} - f; \\
0 & \text{otherwise}.
\end{cases}
\]
Clearly \( s^\lambda \) is in \( R_f(V) \) for every integer \( \lambda \geq 0 \), and \( \lim_{\lambda \to +\infty} \Psi(s^\lambda) = -\infty \), a contradiction. Since \( \sum_{r \in V} \psi(r)s_r \geq \alpha \) is non-trivial, this shows that \( \alpha > 0 \).

Therefore, every nontrivial valid inequality for \( \text{conv}(R_f(V)) \) can be written in the form
\[
\sum_{r \in V} \psi(r)s_r \geq 1.
\]
When \( \psi \) is minimal and non-negative, \( B_\psi \) is a maximal lattice-free convex body [16] and Observation 2 tells us that \( f \in \text{int}_V B_\psi \).

Moreover, when we consider a maximal lattice-free convex body \( B \) in a lattice-subspace, we have an interpretation of the function \( \psi_B \) defined in the previous subsection as follows. The function \( \psi_B \) can be defined by
\[
\psi_B(r) = \inf_{\lambda > 0} \{ \lambda^{-1} : f + \lambda r \in B \}, \quad r \in V.
\]
We define a set \( \mathcal{M}_V \subset \mathcal{B}_V \) as
\[
\mathcal{M}_V = \{ B \subseteq V : B \text{ is a maximal lattice-free convex body of } V \text{ such that } f \in \text{int}_V B \}.
\]
Then Theorem 12 implies the following in rational subspaces.
Corollary 15. Let $\psi : V \to \mathbb{R}$. The function $\psi$ is a minimal valid function for $R_f(V)$ if and only if $\psi = \psi_B$ for some $B \in \mathcal{M}_V$.

And finally, as a special case of Theorem 13 we have that

Corollary 16.

$$\text{conv}(R_f(V)) = \left\{ s \in V : \sum_{r \in V} \psi_B(r)s_r \geq 1 \quad B \in \mathcal{M}_V \right\}.$$  

2.1.3 Relation between $R_f$ and $R_f(r^1, \ldots, r^k)$

In [12], many connections are described between $R_f$ and $R_f(r^1, \ldots, r^k)$ of which the following relates to extreme valid inequalities. We refer the reader to [12] for the proof.

Theorem 17. Given $B \in \mathcal{M}_V$, let $L = \text{lin}(B)$ and let $P = B \cap (f + L^\perp)$. Then $B = P + L$, $L$ is a rational space, and $P$ is a polytope. Let $v_1, \ldots, v_k$ be the vertices of $P$, and $r_{k+1}, \ldots, r_{k+h}$ be an integral basis of $L$. Let $r^j = v^j - f$ for $j = 1, \ldots, k$.

Then $\sum_{r \in V} \psi_B(r)s_r \geq 1$ is an extreme inequality for $\text{conv}(R_f(V))$ if and only if $\sum_{j=1}^{k} s_j \geq 1$ is an extreme inequality for $\text{conv}(R_f(r^1, \ldots, r_{k+h}))$.

Other connections, such as the details of how $R_f(r^1, \ldots, r^k)$ can be viewed as a face of $R_f$ are established in [12] and also how valid inequalities for $R_f(r^1, \ldots, r^k)$ also correspond to maximal lattice-free sets (we discussed this correspondence for $R_f$ above). Topological properties, polyhedrality and dimensionality issues for $R_f(r^1, \ldots, r^k)$ are also discussed in the same paper.

3 Relative Strength of Two Row Cuts

A classical result of Meyer [39] states that if the rays $r^1, \ldots, r^k$ are all rational then $R_f(r^1, \ldots, r^k)$ is a polyhedron. In the case of $m = 2$, Andersen et al. [1], Cornuejols and Margot [20] characterized the facets of this polyhedron in terms of maximal lattice-free sets in 2 dimensions. A consequence of Theorem 8 is the following.

Theorem 18. (Lovász [38]) In the plane, a maximal lattice-free convex set with nonempty interior is one of the following:

(i) A split $c \leq ax_1 + bx_2 \leq c + 1$ where $a$ and $b$ are coprime integers and $c$ is an integer;

(ii) A triangle with an integral point in the interior of each of its edges;

(iii) A quadrilateral containing exactly four integral points, with exactly one of them in the interior of each of its edges; Moreover, these four integral points are vertices of a parallelogram of area 1.

In fact, an even more complete characterization of maximal lattice-free sets in two dimensions can be obtained. Figure 2 illustrates all the different possibilities.

We again state the system governing $R_f(r^1, \ldots, r^k)$ in two dimensions below.

13
Type 1  Type 2  Type 3  Quadrilateral  Split

Figure 2: Maximal lattice-free convex sets with nonempty interior in $\mathbb{R}^2$

$$x = f + \sum_{i=1}^{k} r^i s_i$$
$$x \in \mathbb{Z}^2$$
$$s \geq 0.$$

3.1 Motivation

A natural question to ask is how well do the inequalities derived from splits, triangles and quadrilaterals approximate the integer hull. An unbounded maximal lattice-free convex set in two dimensions is called a split. It has two parallel edges whose direction is called the direction of the split. Split inequalities for (7) are valid inequalities that can be derived by combining the two equations in (7) and by using the integrality of $\pi_1 x_1 + \pi_2 x_2$, where $\pi \in \mathbb{Z}^2$ is normal to the direction of the split. Similarly, for general $m$, the equations can be combined into a single equality from which a split inequality is derived. Split inequalities are equivalent to Gomory mixed integer cuts [40]. Empirical evidence shows that split inequalities can be effective for strengthening the linear programming relaxation of MILPs [9, 23]. Interestingly, triangle and quadrilateral inequalities cannot be derived from a single equation [1]. They can only be derived without aggregating the two equations. Recent computational experiments by Espinoza [26] indicate that quadrilaterals also induce effective cutting planes in the context of solving general MILPs. In [11], we consider the relative strength of split, triangle and quadrilateral inequalities for $R_f(r^1, \ldots, r^k)$ from a theoretical point of view. We use an approach for measuring strength initiated by Goemans [27], based on the following definition.

Let $Q \subseteq \mathbb{R}_+^n \setminus \{0\}$ be a polyhedron of the form $Q = \{x : a^i x \geq b_i \text{ for } i = 1, \ldots, m\}$ where $a^i \geq 0$ and $b_i \geq 0$ for $i = 1, \ldots, m$ and let $\alpha > 0$ be a scalar. We define the polyhedron $\alpha Q$ as $\{x : \alpha a^i x \geq \alpha b_i \text{ for } i = 1, \ldots, m\}$. Note that $\alpha Q$ contains $Q$ when $\alpha \geq 1$. It will be convenient to define $\alpha Q$ to be $\mathbb{R}_+^n$ when $\alpha = +\infty$. We now state the results that we obtain in [11] in the next subsection.

3.2 Results

Let the split closure $S_f(r^1, \ldots, r^k)$ be the intersection of all split inequalities, let the triangle closure $T_f(r^1, \ldots, r^k)$ be the intersection of all inequalities arising from maximal lattice-free triangles, and let the quadrilateral closure $Q_f(r^1, \ldots, r^k)$ be the intersection of all inequalities
arising from maximal lattice-free quadrilaterals. Since all the facets of $R_f(r_1, \ldots, r_k)$ are
induced by these three families of maximal lattice-free convex sets, we have

$$R_f(r_1, \ldots, r_k) = S_f(r_1, \ldots, r_k) \cap T_f(r_1, \ldots, r_k) \cap Q_f(r_1, \ldots, r_k).$$

It is known that the split closure is a polyhedron (Cook, Kannan and Schrijver [19]) but such
a result is not known for the triangle closure and the quadrilateral closure. In [11] we show
the following results.

**Theorem 19.** $T_f(r_1, \ldots, r_k) \subseteq S_f(r_1, \ldots, r_k)$ and $Q_f(r_1, \ldots, r_k) \subseteq S_f(r_1, \ldots, r_k)$.

This theorem may seem counter-intuitive because some split inequalities are facets of
$R_f(r_1, \ldots, r_k)$ [1, 21]. However, we show that any split inequality can be obtained as the
limit of an infinite collection of triangle inequalities. In the proof of Theorem 19 we show that
if a point is cut off by a split inequality, then it is also cut off by some triangle inequality.
Consequently, the intersection of all triangle inequalities is contained in the split closure.
The same is true of the quadrilateral closure.

![Figure 3: Illustration for Example 20](image)

**Example 20.** As an illustration, consider the simple example in Figure 3. The split inequality
is $s_1 \geq 1$ and the split closure is given by $\{(s_1, s_2) \mid s_1 \geq 1, s_2 \geq 0\}$. All triangle inequalities
are of the form $as_1 + bs_2 \geq 1$ with $a \geq 1$ and $b > 0$. The depicted triangle inequality is of the
form $s_1 + 2s_2 \geq 1$ and by moving the corner $v$ closer to the boundary of the split, one can get
a triangle inequality of the form $s_1 + bs_2 \geq 1$ with $b$ tending to 0, but remaining positive. Note
that the split inequality can not be obtained as a positive combination of triangle inequalities,
but any point cut by the split inequality is cut by one of the triangle inequalities.

We further study the strength of the triangle closure and quadrilateral closure in the sense
defined in Section 3.1. We show that both the triangle closure and the quadrilateral closure
are good approximations of the integer hull $R_f(r_1, \ldots, r_k)$ in the sense that
Theorem 21.
\[ R_f(r^1, \ldots, r^k) \subseteq T_f(r^1, \ldots, r^k) \subseteq 2R_f(r^1, \ldots, r^k) \quad \text{and} \]
\[ R_f(r^1, \ldots, r^k) \subseteq Q_f(r^1, \ldots, r^k) \subseteq 2R_f(r^1, \ldots, r^k). \]

Finally we show that the split closure may not be a good approximation of the integer hull.

Theorem 22. For any \( \alpha > 1 \), there is a choice of \( f, r^1, \ldots, r^k \) such that
\[ S_f(r^1, \ldots, r^k) \not\subseteq \alpha R_f(r^1, \ldots, r^k). \]

Some of the above results have been generalized by Andersen, Wagner and Weismantel to the problem with an arbitrary number of rows [3]. These results provide additional support for the recent interest in cuts derived from two or more rows of an integer program [1, 2, 16, 21, 25, 26, 28].

4 The Infinite Group Problem with One Constraint

We now describe some results related to the facets of the infinite group problem. In Mathematical Programming 2003, Gomory and Johnson [32] conjecture that the facets of the infinite group problem with one equality constraint are always generated by piecewise linear functions. That is, the valid functions corresponding to facets for \( G_f \) with only one constraint are piecewise linear. In [13] we give an example showing that the Gomory-Johnson conjecture is false.

To reiterate, the infinite group problem with one constraint is the following.

\[
    f + \sum_{r \in [0,1]} r s_r \in \mathbb{Z}
\]
\[
    s_r \in \mathbb{Z}_+ \quad \forall r \in [0,1]
\]
\[
    s \text{ has finite support,} \quad (8)
\]

where additions are performed modulo 1.

Minimal valid functions for \( G_f \) are subadditive [32] but need not be positively homogeneous and hence we cannot hope for a characterization like Lemma 10 in this case. In fact, they need not even be continuous as shown by Dey et al. [24] where they exhibit extreme inequalities which are discontinuous. However, if we concentrate on continuous valid functions, the following conjecture was made by Gomory and Johnson in [32] about facets for (8) which are continuous.

Conjecture 23 (Facet Conjecture). If \( \pi \) is a facet for (8), then \( \pi \) is piecewise linear.

We show that the above conjecture is false by exhibiting a facet for (8) which is not piecewise linear. This is in contrast with Theorem 9 and Corollary 15 which imply that the extreme functions for the semi-infinite relaxation are always piecewise linear, because maximal lattice-free convex sets are always polyhedra according to Theorem 7. In the next section, we give the construction of a facet for (8) which is not piecewise linear.
4.1 The construction

We first define a sequence of valid functions \( \psi_i : [0,1) \rightarrow \mathbb{R} \) that are piecewise linear, and then consider the limit \( \psi \) of this sequence. We will then show that \( \psi \) is a facet but not piecewise linear.

Let \( 0 < \alpha < 1 \). \( \psi_0 \) is the triangular function given by

\[
\psi_0(x) = \begin{cases} 
\frac{1}{\alpha} x & 0 \leq x \leq \alpha \\
\frac{1}{1-\alpha} x & \alpha \leq x < 1.
\end{cases}
\]

(Notice that the corresponding inequality \( \sum_{r \in [0,1)} \psi_0(r) s_r \geq 1 \) defines the Gomory mixed-integer inequality.)

![Figure 4: First two steps in the construction of the limit function](image)

We first fix a nonincreasing sequence of positive real numbers \( \epsilon_i \), for \( i = 1, 2, 3, \ldots \), such that \( \epsilon_1 \leq 1 - \alpha \) and

\[
\sum_{i=1}^{+\infty} 2^{i-1} \epsilon_i < \alpha.
\]  

(9)

For example, \( \epsilon_i = \alpha \left( \frac{1}{2} \right)^i \) is such a sequence when \( 0 < \alpha \leq \frac{4}{5} \). The upper bound of \( \frac{4}{5} \) for \( \alpha \) is implied by the fact that \( \epsilon_1 \leq 1 - \alpha \).

We construct \( \psi_{i+1} \) from \( \psi_i \) by modifying each segment with positive slope in the graph of \( \psi_i \) as follows.

For every maximal (with respect to set inclusion) interval \( [a, b] \subseteq [0, \alpha] \) where \( \psi_i \) has constant positive slope we replace the line segment from \((a, \psi_i(a))\) to \((b, \psi_i(b))\) with the following three segments.

- The segment connecting \((a, \psi_i(a))\) and \( (\frac{a+b-\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)}) \),
- The segment connecting \( (\frac{a+b-\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)}) \) and \( (\frac{a+b+\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)}) \),
- The segment connecting \( (\frac{a+b+\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)}) \) and \((b, \psi_i(b))\).
Figure 4 shows the transformation of $\psi_0$ to $\psi_1$ and $\psi_1$ to $\psi_2$.

The function $\psi$ which we show to be a facet but not piecewise linear is defined as the limit of this sequence of functions, namely

$$\psi(x) = \lim_{i \to \infty} \psi_i(x)$$

(10)

In [13] we show that each function $\psi_i$ is well defined and is a facet. Moreover, we analyze the limit function $\psi$, showing that it is well defined (that is, the above limit exists), is a facet, but is not piecewise linear.

5 Proposed Work

5.1 Analyzing Facets for $G_f$ using Maximal Lattice-free sets

Dey and Wolsey [25] suggest techniques for getting minimal and extreme inequalities for $M_f$ using maximal lattice-free convex sets. We use a similar idea to interpret the facets for $G_f$ described in [32] using maximal lattice-free sets. We illustrate the idea on an example.

We consider the problem $G_f$ with a single equality constraint and $f = (\frac{1}{2}, 2)$. Consider the triangle $T$ in two dimensions with the following vertices: $(-1, 1/2)$, $(-1, 3/2)$ and $(3, -1/2)$. Let $\bar{f} = (0, 1/2)$. For any ray $r \in \mathbb{R}^2$, define

$$\psi_T(r) = \inf_{\lambda > 0} \{\lambda^{-1} : f + \lambda r \in T\}.$$ 

Note that this is exactly how the minimal valid function corresponding to the triangle is defined for the problem $R \bar{f}$ in Section 2.1.2.

For any $r \in [0, 1]^m$, let $\phi_{GMI}(r) = \min \{\psi_T((r, 0)), \psi_T(r, 1)\}$. It can be checked that $\phi_{GMI}$ is simply a two-slope facet for the single constraint infinite group problem described in [32]. See Figure 5.

The above idea can be generalized to obtain other two slope facets from an appropriate triangle in two dimensions. This idea of “shifting” rays modulo the lattice and picking the best coefficient is also exactly the same idea behind the lifting technique described in Dey and Wolsey [25]. We describe the idea in general for $G_f$ with an arbitrary number of rows, say $m$ as described in the Introduction. We wish to use maximal lattice-free sets in $m + c$ dimensions to obtain valid inequalities for $G_f$ with $c \geq 1$.

Consider the set $G_f$ for some $f \in \mathbb{R}^m$. Define $\tilde{f} \in \mathbb{R}^{m+c}$ to be $(f, 0)$ which is the vector with 0’s in the extra c dimensions. Pick a maximal lattice-free convex set $B$ in $\mathbb{R}^{m+c}$, such that $\tilde{f} \in \text{int}(B)$. For any $\tilde{r} \in \mathbb{R}^{m+c}$, let $\psi_B(\tilde{r}) = \inf_{\lambda > 0} \{\lambda^{-1} : f + \lambda \tilde{r} \in B\}$ as in Section 2.1.2. Now for any $r \in [0, 1]^m$ from the problem $G_f$, define

$$\phi_B(r) = \min_{w \in \mathbb{Z}^m, u \in \mathbb{Z}^c} \psi_B\left(\frac{(r + w)}{u}\right).$$

(11)

Claim 1. $\sum_{r \in [0, 1]^m} \phi_B(r)s_r \geq 1$ is valid for $G_f$.
Proof. The idea is to aggregate the variables who achieve the minimum in the same “dimension”. For a ray $r \in [0,1]^m$, let $w^r$ and $v^r$ be the solution to (11). Then the following is a relaxation of $G_f$.

$$
\begin{align*}
  x &= f + \sum_{r \in [0,1]^m} (r + w^r) s_r \\
  y_i &= \sum_{r \in S_i} v_i^r s_r & \text{for } i = 1, 2, \ldots, c \\
  x &\in \mathbb{Z}^m \\
  y_i &\in \mathbb{Z} \\
  s_r &\geq 0
\end{align*}
$$

By Corollary 15, we know that $\sum_{r \in [0,1]^m} \phi_B(r) s_r \geq 1$ is valid for (12) and hence is valid for $G_f$.

**Question 1:** Consider the case when $m = 1$ (single constraint problem). Let $c = 1$ (so we will consider two-dimensional lattice-free sets to “lift”). The results of Dey and Wolsey[25] can be used to characterize which maximal lattice-free sets in two dimensions give rise to minimal inequalities for the single constraint $G_f$. For which two dimensional sets are they extremal? Is there a “geometric” proof when they are not extremal?

In this context, one can make the following observation about the finite group problem. As before, the finite group problem refers to the following problem.
\[ f + \sum_{i=1}^{k} r_i s_i \in \mathbb{Z}^m, s_i \in \mathbb{Z}, s_i \geq 0 \quad \text{for all } i \quad (13) \]

where we assume \( r_i \in [0, 1]^m \) for all \( i \).

**Fact 24.** We consider the finite group problem with \( m = 1 \) (single constraint problem) and \( c = 1 \). Consider all maximal lattice-free sets \( B \in \mathbb{R}^2 \). For every such set, we do the “lifting” as described above and derive a valid inequality \( \sum_{i=1}^{k} \phi_B(r_i)s_i \geq 1 \). Let \( S = \bigcap_{\text{maximal lattice-free sets } B} \sum_{i=1}^{k} \phi_B(r_i)s_i \geq 1 \). Then

\[ S = \bigcap_{w \in \mathbb{Z}^k, v \in \mathbb{Z}^k} \text{conv} \left\{ \begin{array}{l}
 x = f + \sum_{i=1}^{k} (r_i + w_i)s_i \\
 y = \sum_{i=1}^{k} v_is_i \\
 x \in \mathbb{Z}, y \in \mathbb{Z}, s_r \geq 0
\end{array} \right\} \quad (14) \]

**Proof.** Let \( C = \bigcap_{w \in \mathbb{Z}^k, v \in \mathbb{Z}^k} \text{conv} \left\{ \begin{array}{l}
 x = f + \sum_{i=1}^{k} (r_i + w_i)s_i \\
 y = \sum_{i=1}^{k} v_is_i \\
 x \in \mathbb{Z}, y \in \mathbb{Z}, s_r \geq 0
\end{array} \right\} \).

Any inequality \( \phi_B \) defining \( S \) is clearly valid for \( C \), since \( \psi \) is valid for the relaxation constructed in the proofs above. So \( C \subseteq S \).

Consider some \( w \in \mathbb{Z}^k, v \in \mathbb{Z}^k \). Define \( C_{w,v} \) as

\[ \text{conv} \left\{ \begin{array}{l}
 x = f + \sum_{i=1}^{k} (r_i + w_i)s_i \\
 y = \sum_{i=1}^{k} v_is_i \\
 x \in \mathbb{Z}, y \in \mathbb{Z}, s_r \geq 0
\end{array} \right\}. \]

Any extreme valid inequality \( \phi' \) for \( C_{w,v} \) comes from some maximal lattice-free set \( B \) in \( \mathbb{R}^{m+c} \) by Corollary 15. We claim that \( \phi_B \) that we obtain during the “lifting” dominates this inequality. This is because, the coefficient of any ray \( r_i \) is only smaller in \( \phi_B \) since we picked the best \( w_r, v_r \). This proves \( S \subseteq C \).

We now show that the split closure of the finite group problem with one constraint can be obtained in this manner.

**Proposition 25.** Consider the polyhedron

\[ P = \left\{ (x, s) : x = f + \sum_{j \in J} r_j s_j, s \geq 0 \right\} \]

Let \( S(P) \) be the split closure of \( P \) (when we impose integrality constraints on \( x, s \)). Then \( C \subseteq S(P) \), where \( C \) is as defined in the proof above.
Proof. Any split is of the form $\alpha x + \sum_{j \in J} v_j s_j \leq n$ OR $\alpha x + \sum_{j \in J} v_j s_j \geq n + 1$, where $\alpha \in \mathbb{Z}^m$, $v_j, n \in \mathbb{Z}$.

The condition $\alpha x + \sum_{j \in J} v_j s_j \in \mathbb{Z}$ is at least as strong, which is what we enforce in $C$.

**Question 2:** Are there examples where the containment $C \subseteq S(P)$ is strict?

Let us return to the general setting for this approach with $m \geq 1$ and $c \geq 1$. Consider all maximal lattice-free sets $B \in \mathbb{R}^{k+c}$. For every such set, we do the “lifting” as described above and derive a valid inequality $\sum_r \psi_B(r)s_r \geq 1$. Let $S = \bigcap_{\text{maximal lattice-free sets }B} \psi_B(r)s_r \geq 1$.

**Question 3:** How does the set $S$ compare to $\text{conv}(G_f)$? In other words, can we get the convex hull of all solutions for the infinite group problem with inequalities derived in such a manner from maximal lattice-free sets in higher dimensions? It is conceivable to use all values of $c \geq 1$ and use the maximal lattice sets from all those dimensions - can we now get $\text{conv}(G_f)$? The answer to this is in the negative. This is because the valid functions that we would obtain would all be piecewise linear, whereas we have exhibited in [13] that there are facets which are not piecewise linear. However, if we restrict our attention to piecewise linear valid functions - can they all be derived from a maximal lattice-free set in $\mathbb{R}^{m+c}$ for some $c \geq 1$?

**Question 4:** A more modest question is to ask whether all two-slope facets for the single constraint infinite group problem can be obtained from maximal lattice-free triangles in this manner?

5.2 The Algorithmic Perspective : Optimizing over Corner Polyhedra

We concentrate on the finite problems in this section with a motivation to provide provable algorithms for solving them. Usually the sets $R_f(r^1, \ldots, r^k)$, $G_f(r^1, \ldots, r^k)$ and corner polyhedra (defined by a system like (3)) will have a huge number of extreme and minimal inequalities. It is unreasonable to add all these inequalities while using a branch and cut approach to solving these problems. It is essential to characterize the families of valid inequalities which are “good” and obtain separation oracles for such families of inequalities.

To illustrate with an example, the Chvatal closure of the matching polytope immediately gives the integer hull and there is a polynomial time algorithm to separate over this exponential-sized family of cutting planes. This immediately implies a polynomial algorithm for the matching problem [34]. We wish to obtain results in the same spirit. We propose a two-pronged approach in this direction as outlined in the two following subsections.

5.2.1 Strength of Corner Polyhedra and its variants

We want to study how close we get to the integer hull for a problem using the corner polyhedra or a subset of its valid inequalities. The results outlined in Section 3.2 are in this direction for cutting planes obtained from maximal lattice-free triangles and quadrilaterals. Similar results have also been obtained by Andersen, Wagner and Weismantel for problems with more than two rows [3]. On the negative side, it has been shown in [17] that in the Stable Set (Independent Set, Vertex packing) problem on graphs, the corner polyhedra are only as strong as the $\{0, \frac{1}{2}\}$-Chvatal closure.
Question 1: For a given family of valid inequalities for the corner polyhedra or its relaxation $R_f(r_1, \ldots, r_k)$, say the triangle inequalities, can we identify specific classes of problems where we provably approximate the integer hull well? The Goemans measure of approximation outlined in Section 3.2 makes particular sense from an approximation algorithms point of view because this implies exactly the same upper bound on the integrality gap of the relaxation obtained with this family of inequalities.

Question 2: Negative results in the spirit of [17] are also of relevance. Can we identify problems where a particular class of inequalities obtained from corner polyhedra add no extra value? Such a research direction would provide more insight and intuition into what kind of problems are amenable to good algorithms using corner polyhedra.

Question 3: Can we obtain any kind of interesting results about how well corner polyhedra approximate general MILPs?

5.2.2 Optimizing over Corner Polyhedra and its variants

Knowledge of good families of inequalities as proposed in the previous subsection needs a counterpart in efficient algorithms to optimize over these inequalities. For the set $R_f(r_1, \ldots, r_k)$ with a fixed number of rows, we have a polynomial time algorithm for optimizing over it. This result is formally stated in the theorem below. The importance of such results is in the following simple observation. Suppose we have an efficient algorithm for optimizing over the intersection of all the corner polyhedra corresponding to every feasible basis, combined with a result which guarantees a very good approximation of the integer hull by the corner polyhedra. This could form the basis for good approximation algorithms because the integrality gap is small.

Theorem 26. Define

$$z(f) = \begin{cases} \min \sum_{i=1}^{k} c_i s_i \\ x = f + \sum_{i=1}^{k} r_i s_i \\ x \in \mathbb{Z}^d, s_i \geq 0 \end{cases}$$

Then $z(f)$ is achieved by a solution where only $d$ of the $s_i$ variables are non-zero. Moreover, $z(f)$ can be computed in polynomial time.

Proof. First note that the objective function can be assumed to be the all 1’s vector by a suitable scaling of the rays (replace each ray $r_i$ by the ray $\frac{1}{s_i} r_i$ and let $s'_i = c_i s_i$).

Let $C(\alpha)$ be the convex hull of the points $\{\alpha r_1, \alpha r_2, \ldots, \alpha r_k\}$. Find the largest $\alpha$ such that $C(\alpha)$ is lattice-free. Let this be $\alpha^*$.

We claim $\alpha^* \leq z(f)$. Consider $C(\alpha^*)$. This is contained in some maximal lattice-free set $B^*$. The inequality $\psi_{B^*}$ derived from $B^*$ is valid for the problem. Moreover, for any ray $r_i$, $\psi_{B^*}(r_i) \leq \frac{1}{\delta}$. This implies that for any point $\bar{s}$ such that $\sum_{i=1}^{k} \psi_{B^*}(r_i) \bar{s}_i \geq 1$ we have that $\sum_{i=1}^{k} s_i \geq \alpha^*$. Therefore every feasible solution has value at least $\alpha^*$ and so $\alpha^* \leq z(f)$.

We now show that $\alpha^* \geq z(f)$, by exhibiting a feasible solution with value $\alpha^*$. Consider $C(\alpha^*)$. Some facet of this polytope has an integer point $p$ in it (since $C(\alpha^*)$ was maximally dilated). This facet is of dimension at most $d - 1$, and Caratheodory’s theorem says that $p$ is the convex hull of at most $d$ vertices of this facet. So $p = \sum_{j \in J} \lambda_j (f + \alpha^* r_i)$, for some subset
\[ J \subseteq \{1, 2, \ldots, k\} \text{ with } |J| \leq d \text{ and } \sum_{j \in J} \lambda_j = 1, \lambda_j \geq 0. \] Therefore, setting \( s_j = \alpha^* \lambda_j \) for \( j \in J \) and \( s_j = 0 \) otherwise, we get a feasible solution with value \( \alpha^* \). Note that at most \( d \) of the variables are non-zero in this feasible solution.

Note that \( \alpha^* \) can be computed in polynomial time when \( d \) is fixed. This follows from the fact that convex hulls can be computed in polynomial time for fixed \( d \) and we can check if the scaled convex hull is lattice-free by integer programming in fixed \( d \) dimensions. Using a binary search on the scaling factor of the convex hull, we can find \( \alpha^* \).

**Question**: Given a family of valid inequalities, say triangle inequalities, can we obtain good separation oracles for (or equivalently, efficient algorithms to optimize over) these inequalities? We want to investigate for which family of inequalities suggested by the study of corner polyhedra and its variants like \( R_f(r^1, \ldots, r^k) \) and \( G_f(r^1, \ldots, r^k) \) can we have such efficient algorithms.

### 5.3 Study of Maximal Lattice-free sets in 3 dimensions

In [14], we obtain complete characterizations of maximal lattice-free sets in three dimensions. The results point to the fact that the number of possibilities for maximal lattice-free sets grow extremely large when one moves from two dimensions to three dimensions. Moreover, all three dimensional polytopes correspond to 3-connected simple, planar graphs by Steinitz Theorem [42]. Motivated by the results in [14], we would like to settle the following conjecture.

**Conjecture 27.** Maximal Lattice-free convex polytopes in three dimensions correspond to 3-connected, simple planar graphs with at most 8 faces. More precisely, for every 3-connected, simple planar graph \( G \) with at most 8 faces there exists a maximal lattice-free polytope in 3 dimensions whose skeleton is \( G \).

### 5.4 Computational and Empirical Work

We also wish to provide substantial evidence to the effectiveness of the theory behind corner polyhedra and maximal lattice-free convex sets by solving actual problems from MIPLIB. The results in [11] suggest triangle inequalities which do significantly better than the split closure. In particular, the proof of Theorem 22 is constructive and gives an example of an integer program and a triangle inequality for it which does much better than the split closure. Motivated by these examples, we have implemented code to generate such inequalities for general MILPs and are currently in the process of testing their effectiveness on problems from MIPLIB. A main goal is to find out if such inequalities provide some additional advantage to solving MILPs along with traditional cutting planes. Moreover, such experiments would give some empirical hints to answering questions outlined in Section 5.2.1. This work is being undertaken in collaboration with Pierre Bonami, Gerard Cornuejols and Francois Margot.

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References


