The Effect of Model Error on the Valuation and Hedging of Natural Gas Storage

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Abstract

How should price models be chosen for real option valuation and how should real option cash flows be hedged in the presence of model error? We consider a specific type of model error in the context of the natural gas storage real option: The assumed number of factors in a multifactor futures price model differs from the true number of factors. In terms of valuation, model error does not seem to be significant. Specifically, the valuations obtained with a family of empirically calibrated seasonal multifactor models of the futures curve are comparable as long as the most important three to six statistical factors are included. In terms of hedging, there are many ways to delta hedge using different combinations of futures contracts. Theoretically, these different hedging approaches are all equivalent without model error. However, they are not equivalent in the presence of model error. In particular, bucket hedging outperforms a naïve implementation of factor hedging, the performance of which can be disastrous. Enhanced factor hedging, in which the particular futures contracts traded are optimized, also performs well provided that a sufficient number of factors are used. Bucket hedging is remarkably robust to model error irrespective of the number of factors assumed, and is easier to implement than enhanced factor hedging. Our findings demonstrate the differential impact of model error on the valuation and hedging of the real option to store natural gas, and have potential relevance beyond this specific application.

1. Introduction

Storage allows merchants to trade a commodity physically at different prices over time. Contracts for natural gas storage capacity located at, or connected to, natural gas market hubs thus represent real options on natural gas prices. The availability of natural gas futures and options on futures, traded on markets such as the New York Mercantile Exchange (NYMEX), the IntercontinentalExchange (ICE), and the European Energy Exchange, makes possible the valuation and hedging of these real options using modern financial replication techniques. Effectively managing natural gas storage requires a stochastic model of the risk neutral natural gas price dynamics, optimization of the inventory policy, and the computation of risk sensitivities to dynamically hedge changes in the value of the storage option.

The choice of the stochastic model for natural gas prices plays a key role in valuation and hedging. According to authors in the practice-based literature, such as Maragos [34, pp. 440,
449-453], Eydeland and Wolyniec [20, pp. 351-367], and Gray and Khandelwal [27, 28], the preferred stochastic modeling approach among traders for managing the monthly volatility component of natural gas storage is a multifactor futures curve model (Clewlow and Strickland [16, Chapter 8], Seppi [43], Eydeland and Wolyniec [20, Chapter 5], Geman [23, Chapter 3]). However, these models inevitably embed model error; that is, they potentially disagree with the real evolution of natural gas prices.

In this paper, we study how to choose a multifactor futures price model for valuation of the real option to store natural gas and how to hedge the cash flows of this real option in the presence of model error (Hull [30, Chapter 20]). The particular type of model error we consider occurs because the assumed number of price factors differs from the true number of factors.

Several multifactor models of risk neutral commodity futures curve dynamics have been proposed in the literature. Eydeland and Wolyniec [20, pp. 351-367] and Gray and Khandelwal [27, 28] discuss modeling the full term structure of the natural gas futures curve using as many factors as there are futures delivery dates over the term of a storage contract. This approach is related to the so-called string and BGM models used to value fixed income options (Kennedy [31], Brace et al. [11], Longstaff et al. [33]). Alternatively, other authors restrict attention to a few common factors that dominate the statistical dynamics of the futures curve. A typical number of factors is three, which have qualitative interpretations of level (shifting), slope (tilting), and curvature (bending) effects (Cortazar and Schwartz [17], Clewlow and Strickland [16, Chapter 8], Borovkova and Geman [9]). Other numbers of factors have also been discussed (see Borovkova and Geman [8], who use two factors in addition to deterministic seasonality factors, and Bjerksund et al. [3] with six factors).

In this paper, we consider a family of calibrated seasonal multifactor futures curve models (Blanco and Stefiszyn [6]) that nests both full and lower dimensional specifications. We calibrate this family of price models to NYMEX natural gas futures price data from 1997 through 2006. We then use these calibrated models to value and hedge the natural gas storage real option, on both simulated and real price data.

The curse of dimensionality makes computing the optimal inventory policy impractical with most of the specifications of the price model that we consider. Thus, we instead value natural gas storage heuristically by using the rolling intrinsic policy. This policy is widespread among practitioners and yields near optimal valuations when used with a full dimensional extension of the Black [4] model (Lai et al. [32]).

Using this policy, we find that for twenty-four month storage contracts a misspecified model
with the most important three to six statistical factors obtains 98% of the value of storage based on a true model with up to twenty-four empirically calibrated factors. This finding suggests that model error due to omitted factors is not important for valuation purposes, as long as one includes the most important statistical factors in the futures price model.

There are several approaches to delta hedge (Hull [30, Chapter 6]) the cash flows of a natural gas storage contract. With bucket hedging, one trades futures contracts with delivery dates corresponding to each date on which physical trading has a cash flow. For example, bucket hedging a two-year storage contract entails trading in twenty-four futures contracts. Bucket hedging can be applied irrespective of the number of assumed factors in the futures price model. In contrast, factor hedging involves trading a number of futures contracts equal to the number of factors (Cortazar and Schwartz [17], Schwartz [39], Clewlow and Strickland [16, §9.5]). For example, factor hedging a two-year storage contract based on a three-factor model entails trading in just three futures contracts.

We consider bucket hedging and two versions of factor hedging, one naïve and one enhanced. Naïve factor hedging takes positions in futures contracts whose maturities are nearest to the trading date. These positions can be extreme. Our enhanced factor hedging approach chooses futures contracts in a way that leads to less extreme positions in individual futures contracts.

On simulated data without model error, all three hedging approaches display good performance, as expected. With simulated data and model error due to omitted factors, bucket hedging dominates naïve factor hedging, whose performance is mostly disastrous, and also outperforms enhanced factor hedging with five or fewer factors to a lesser extent. On simulated data with model error arising from excess assumed factors, all three hedging methods behave as if there were no model error. On real data, where the true futures price model is unknown, bucket hedging natural gas storage contracts with twelve month terms outperforms enhanced factor hedging with less than three factors. The two methods exhibit similar performance with three or more factors.

Thus, our analysis suggests that the negative effect of model error due to omitted factors on hedging natural gas storage can be effectively dealt with by either trading all the relevant futures contracts, as in bucket hedging, or by carefully choosing which contracts to trade, as in enhanced factor hedging, provided that the price model includes a sufficient number of factors. With factor hedging, careful selection of which contracts to trade is critical. The trading positions in naïve factor hedging can be very large, which makes it extremely sensitive to model error with omitted factors. Avoiding this through enhanced factor hedging entails additional
computation in the selection of the hedging contracts. In contrast, bucket hedging appears remarkably robust to model error, requires no additional computation, and is straightforward to implement.

Our findings improve our understanding of the differential effect of model error on the valuation and hedging of natural gas storage and, by extension, the storage of other commodities. They also may inform current practice on the real option management of natural gas storage. In particular, due to its simplicity, practitioners are likely to favor bucket hedging to enhanced factor hedging. Some practitioners prefer full dimensional over lower dimensional factor models. However, bucket hedging does not require a full dimensional model. Our work suggests that the performance of bucket hedging is comparable whether it is based on a full or lower dimensional model. Thus, part of current practice may exhibit redundancy in the number of factors used for valuation, albeit not in the number of contracts used for hedging.

Our analysis also makes methodological contributions. We derive pathwise expressions (Fu and Hu [22], Broadie and Glasserman [12], Boyle et al. [10], Glasserman [26, §7.2]) for the deltas associated with the optimal inventory policy. We heuristically apply these expressions when using the rolling intrinsic policy. This makes feasible our computational analysis. Further, we characterize the trading positions of any factor hedging approach as the sum of a specific delta and a linear combination of additional deltas. The coefficients of this linear combination are inversely proportional to the determinant of a specific matrix whose inverse is used to compute them. This characterization plays a central role in enhanced factor hedging.

The rest of this paper is organized as follows. We review the relevant literature in §2. We present the family of futures price models used in this research in §3. We discuss the exact valuation and hedging of natural gas storage in §4 and §5, respectively. We conduct our model error analysis in §6. We conclude in §7. An Online Appendix includes proofs and examples.

2. Literature Review

Previous research on multifactor models of commodity prices focuses on the ability of different models to explain the statistical variability in price time-series data (Cortazar and Schwartz [17], Gibson and Schwartz [25], Schwartz [39], Routledge et al. [38], Schwartz and Smith [40], Blanco et al. [5], Casassus and Collin-Dufresne [14], Geman and Nguyen [24], Borovkova and Geman [8], and Suenaga et al. [44]). In contrast, we focus on the valuation and hedging performance of a family of multifactor models in the context of a specific real option.
Other authors have studied the valuation of natural gas storage (Chen and Forsyth [15], Boogert and de Jong [7], Lai et al. [32], Thompson et al. [45], Carmona and Ludkovski [13], Secomandi [41], and Wu et al. [46]). None of these papers considers the dependence of the valuation of storage on the number of factors in the price model and how the number of price model factors impacts the performance of different hedging approaches. Bjerksund et al. [3] emphasize the importance of the choice of price model when valuing natural gas storage, but do not carry out the hedging analysis that we perform in this paper. Their analysis suggests that six factors are adequate for valuing natural gas storage. Our analysis suggests that a similar number of factors is adequate both for valuation and hedging of natural gas storage, using bucket or enhanced factor hedging.

Longstaff et al. [33], Fan et al. [21], and Driessen et al. [19] investigate the pricing and hedging of interest rate caps and swaptions using single and multifactor models. Their work illustrates the importance of using multifactor models to hedge these contracts. In particular, Driessen et al. [19] show that bucket hedging outperforms factor hedging. We confirm this result in a different context, natural gas storage, which features inventory, a component notably absent in the fixed income options considered by these authors. Our analysis shows that the futures maturities used with factor hedging must be chosen with care. Moreover, while caps and swaptions are marked-to-market, natural gas storage is not. Thus, our analysis focuses on the ability of different futures hedging approaches to replicate cash flows from physical trading, rather than on hedging market price changes.

We derive pathwise deltas (Fu and Hu [22], Broadie and Glasserman [12], Boyle et al. [10], Glasserman [26, §7.2]) for the storage real option when storage is optimally valued, and use them in a heuristic fashion to delta and factor hedge the value of the storage real option computed by the rolling intrinsic policy. The pathwise approach is rarely used in the real options literature. An exception is Secomandi and Wang [42] who focus on the valuation and hedging of the real option to transport natural gas. Applying the pathwise approach in the natural gas storage case is more involved than in the natural gas transport case, as the storage case requires the analysis of a stochastic dynamic program involving inventory, a fundamentally more complicated model than that used by Secomandi and Wang [42].

Our approach to compute the positions taken by a given factor hedging method reassigns redundant deltas to specific futures contracts. This approach is related to the factor hedging implementation in Driessen et al. [19], but differs from those discussed by Cortazar and Schwartz [17], Schwartz [39], Clewlow and Strickland [16, §9.5], Longstaff et al. [33], and Fan et al. [21],
which are based on a numerical perturbation approach. In addition, we propose a method to improve the selection of the contracts to use with factor hedging, whereas in Longstaff et al. [33], Fan et al. [21], and Driessen et al. [19] they are determined exogenously.

Previous work on the impact of model error on option pricing and hedging includes McEneaney [35], Hobson [29], and Psychoyios and Skiadopoulos [37]. These papers focus on misspecification in a single factor model, market incompleteness, and the hedging performance of different single options. In contrast, we focus on misspecification relating to the number of factors and the number of hedge instruments.

3. A Family of Futures Price Evolution Models

In this section we present a family of $K \in \{1, \ldots, N - 1\}$ factor models to describe the risk-neutral dynamics of the first $N > 1$ contracts in the natural gas futures curve over the term $[T_0 := 0, T_{N-1}]$ of a storage contract. We denote by $F(t, T_m)$ the futures price at time $t \in [0, T_m]$ with maturity $T_m$, $\forall m \in \mathcal{N} := \{0, \ldots, N - 1\}$. The spot price at time $T_m$ is $F(T_m, T_m)$. The dynamics of commodity futures curves are discussed by Cortazar and Schwartz [17], Clewlow and Strickland [16, p. 142], Blanco et al. [5], Eydeland and Wolyniec [20, pp. 201-206], and Geman [23, p. 73] among others. For a fixed number $K$ of factors and by defining $K := \{0, \ldots, K - 1\}$, these dynamics are described by the following stochastic differential equations:

$$\frac{dF(t, T_m)}{F(t, T_m)} = \sum_{k \in \mathcal{K}} \sigma_{m,k}(t) dZ_k(t), \; \forall m \in \mathcal{N} \setminus \{0\}, t \in [0, T_m], \quad (1)$$

where $\sigma_{m,k}(t)$ is the loading coefficient for factor $k$ of the futures price with maturity $T_m$ at time $t$, $dZ_k(t)$ is a standard Brownian motion increment corresponding to factor $k$, and the standard Brownian motion increments corresponding to the different factors have zero instantaneous correlations. This model can capture seasonality in the volatility of futures prices, because each factor loading coefficient depends on the futures trading time. Seasonality in the natural gas price levels is captured by the futures prices themselves.

As in Blanco et al. [5], we impose the restriction that each coefficient $\sigma_{m,k}(\cdot)$ is constant over each time interval $[T_n, T_{n+1})$, with $n \in \mathcal{N}$ and $m \in \mathcal{N}_{n+1} := \{n + 1, \ldots, N - 1\}$. That is, given a factor $k$ and a maturity $T_m$, we assume that $\sigma_{m,k}(\cdot)$ is the constant $\sigma_{m,k,n}$ over the interval $[T_n, T_{n+1})$.

Our empirical calibration of model (1) is based on the factor loading coefficients being the same under the risk neutral and objective futures curve dynamics. Following Blanco et al. [5],
we use a principal component analysis (PCA) to estimate the factor loading coefficients for each month separately, as discussed in §6. Under this specification, given \( n \) and \( m \in \mathbb{N}_{n+1} \), \( t \in [T_n, T_{n+1}] \), and \( t' \in (T_n, T_{n+1}] \), the futures price \( F(t', T_m) \) conditional on \( F(t, T_m) \) is

\[
F(t', T_m) = F(t, T_m) \exp \left[ -\frac{1}{2} (t' - t) \sum_{k \in K} \sigma_{m,k,n}^2 + \sqrt{t' - t} \sum_{k \in K} \sigma_{m,k,n} Y_k \right],
\]

where \( Y := (Y_k, k \in K) \) is a normal random vector with \( K \) components and with mean vector equal to the zero vector and variance-covariance matrix equal to the identity matrix.

In the special case when \( K = N - 1 \) and each loading factor coefficient does not depend on the trading time \( t \), model (1) is equivalent to the multidimensional version of the homoskedastic Black [4] model of futures prices discussed in the practice-based literature (Eydeland and Wolyniec [20, Chapter 5] and Gray and Khandelwal [27, 28]), and used by Lai et al. [32] in the context of the valuation of natural gas storage.

4. Valuation

Natural gas storage contracts give merchants the right to inject, store, and withdraw natural gas at a storage facility over a finite time horizon, subject to capacity and inventory constraints. Capacity constraints specify the maximal amount of natural gas, measured in British thermal units (mmBtus), which a merchant can inject or withdraw per unit of time. Inventory constraints specify the minimal and maximal amounts of natural gas inventory that the merchant can hold at any given point in time. A contract also specifies injections and withdrawal charges, and fuel losses.

We assume that the facility associated with a given storage contract is located nearby a liquid wholesale spot market, where the merchant performs its physical trading activity. This is a realistic assumption for North America, which has roughly one hundred geographically dispersed wholesale markets for natural gas; the National Balancing Point in the United Kingdom; Zebrugge in Belgium; Emden in Germany; and Hub Holland in the Netherlands (Geman [23, Chapter 10]). If the storage facility is not located at a market hub, then transport capacity and cost for the pipeline connecting the storage facility to the relevant hub must also be included.

We also assume that a natural gas futures market is associated with the wholesale physical market. This is realistic for North America, where NYMEX and ICE trade futures contracts with delivery location at Henry Hub, Louisiana, and basis swaps, financially settled forward locational price differences relative to Henry Hub, at about forty other locations. In the United
Kingdom, ICE trades natural futures associated with the National Balancing Point. The European Energy Exchange, located in Germany, also trades natural gas futures.

The rest of this section follows closely the material in Lai et al. [32, §2]. We are interested in valuing the “forward” or “monthly volatility” component of a storage contract, which corresponds to the cash flows associated with making physical natural gas trading decisions on a monthly basis. This is realistic because the spot market in the United States is most liquid during the monthly “bid week” in which blocks of gas for the ensuing month are traded.

The storage valuation problem can be formulated as a Markov decision process (MDP). The timing of the physical trading inventory decisions is assumed to coincide with the futures maturity dates. The stages of the MDP are the futures maturities. Let $a$ denote an action. A withdrawal followed by a sale is modeled as a positive action, a purchase followed by an injection as a negative action, and the do-nothing action as zero. Natural gas purchased (respectively, sold) at time $T_n$ is available (respectively, unavailable) in storage at time $T_{n+1}$.

We let the storage contract minimal and maximal inventory levels be 0 and $x \in \mathbb{R}_+$, respectively. The set of feasible inventory levels is the interval $\mathcal{X} := [0, x]$. The injection and withdrawal capacities per stage are $C^I < 0$ and $C^W > 0$, respectively. The per stage feasible injection action set, withdrawal action set, and action set, respectively, with feasible inventory $x$ are $A^I(x) := [C^I \vee (x - \pi), 0]$, $A^W(x) := [0, x \wedge C^W]$, $A(x) := A^I(x) \cup A^W(x)$, where $\cdot \wedge \cdot \equiv \min\{\cdot, \cdot\}$ and $\cdot \vee \cdot \equiv \max\{\cdot, \cdot\}$.

We use the more compact notation $F_{n,m}$ for futures price in lieu of $F(T_n, T_m)$. We let $F_n := (F_{n,m}, m \in \mathcal{N}_n)$, $\forall n \in \mathcal{N}$, and $F'_n := (F_{n,m}, m \in \mathcal{N}_{n+1})$, $\forall n \in \mathcal{N} \setminus \{N - 1\}$, denote the futures curves at time $T_n$ inclusive and exclusive, respectively, of the spot price at this time, with the conventions $F_N := 0$ and $F'_{N-1} := 0$.

The per stage cash flow associated with the physical trading action $a$ at time $T_n$ is denoted by $p(a, s_n)$, where $s_n \equiv F_{n,n}$ is the spot price at time $T_n$. We model in-kind fuel losses using the coefficients $\phi^W \in (0, 1]$ and $\phi^I \geq 1$ for injections and withdrawals, respectively. The marginal withdrawal and injection costs are $c^W$ and $c^I$, respectively. Thus, the per stage cash flow function given an action $a$ and the spot price $s$ is

$$p(a, s) := \begin{cases} (\phi^I s + c^I) a, & a \in \mathbb{R}_-, \\ 0, & a = 0, \\ (\phi^W s - c^W) a, & a \in \mathbb{R}_+. \end{cases}$$

The value of a storage contract corresponds to the optimal value function in the initial stage and state of the MDP that we now formulate. We denote by $V_n(x_n, F_n)$ the optimal value
function in stage $n$ and state $(x_n, F_n)$ of this MDP. Let $\delta$ denote the constant one-stage risk-free discount factor. Taking advantage of the convention $F_N \equiv 0$ in (3) below, the MDP is

\[
V_N(x_N, F_N) := 0, \quad \forall x_N \in \mathcal{X},
\]

\[
V_n(x_n, F_n) = \max_{a \in \mathcal{A}(x_n)} p(a, s_n) + \delta \mathbb{E} \left[ V_{n+1}(x_n - a, F_{n+1}) | F_n' \right],
\]

\[
\forall n \in \mathcal{N}, (x_n, F_n) \in \mathcal{X} \times \mathbb{R}_+^{N-n}.
\]

In typical applications the number of maturities $N$ is at least twelve, thus the curse of dimensionality makes model (3)-(4) computationally intractable. The unbounded feature of the price related part of the state creates additional difficulties. This makes the heuristic solution of model (3)-(4) an appealing alternative, a topic discussed in §6.

Proposition 1 characterizes the optimal operating policy as having an appealing (stage and price) state-dependent double basestock target structure: In a given stage, do nothing if the inventory is in between the two basestock targets in this stage, otherwise bring the inventory as close as possible to the relevant target. The proof is a simple adaptation of the proofs of Theorem 1 in Lai et al. [32] and Secomandi [41]. These properties are used in §5.

**Proposition 1** (Concavity and basestock optimality). In every stage $n$, the function $V_n(x_n, F_n)$ is concave in $x_n$ for each given $F_n$, and the optimal policy for model (3)-(4) features two basestock targets, $b_n(F_n)$, $\bar{b}_n(F_n) \in \mathcal{X}$, such that $b_n(F_n) \leq \bar{b}_n(F_n)$ and an optimal action $a^*_n(x_n, F_n)$ satisfies

\[
a^*_n(x_n, F_n) = \begin{cases} 
C^I \lor [x_n - b_n(F_n)], & x_n \in [0, b_n(F_n)), \\
0, & x_n \in [b_n(F_n), \bar{b}_n(F_n)], \\
C^W \land [x_n - \bar{b}_n(F_n)], & x_n \in (\bar{b}_n(F_n), \mathcal{X}]. 
\end{cases}
\]

\[
(5)
\]

5. Hedging

In this section, we present exact models for hedging the real option to store natural gas. The value function of model (3)-(4) is defined over a discrete time set. As in theory hedging occurs in continuous time, we map this value function into a continuous time value process. Given a date $t \in [T_{n-1}, T_n]$ for $n \in \mathcal{N} \setminus \{0\}$, we define the following vector of futures prices $G_n(t) := (F(t, T_m), m \in \mathcal{N}_n)$ (at time $t = T_n$ this vector includes the spot price $s_n$). For every $n \in \mathcal{N} \setminus \{0\}$, $t \in [T_{n-1}, T_n]$, and $(x, G_n(t)) \in \mathcal{X} \times \mathbb{R}_+^{N-n}$, we use the function $V_n(\cdot, \cdot)$ to define the continuous time process

\[
U_n(t, x, G_n(t)) := \mathcal{E}(t, T_n) \mathbb{E}[V_n(x, F_n) | G_n(t)],
\]

\[
(6)
\]
where $\delta(t, T_n)$ is the risk-free discount factor from time $T_n$ back to time $t$ ($\delta(T_n-1, T_n) \equiv \delta$). The function $U_n(t, x, G_n(t))$ is the time $t$ updated optimal value of having $x$ inventory units in storage at time $T_n$ given the vector of futures prices at time $t$. Starting in state $(x_{n-1}, F_{n-1})$ at time $T_{n-1}$, we are interested in hedging the changes in the value of $U_n(t, x_{n-1} - a^*_n(x_{n-1}, F_{n-1}), G_n(t))$ due to changes in the futures curve $G_n(t)$ during time interval $(T_{n-1}, T_n]$. We first discuss bucket hedging and factor hedging. Then we discuss the computation of the deltas.

**5.1 Bucket and Factor Hedging**

For every $n \in \mathbb{N} \setminus \{0\}$, we define the time $t \in [T_{n-1}, T_n)$ delta corresponding to maturity $T_m$, with $m \in \mathbb{N}_n$, for $(x, G_n(t)) \in \mathcal{X} \times \mathbb{R}^{N-n}$, as

$$\Delta_{n,m}(t, x, G_n(t)) := \frac{\partial U_n(t, x, G_n(t))}{\partial F(t, T_m)}.$$  

(7)

When $t = T_{n-1}$ we are interested in the partial derivative of $U_n(T_{n-1}, x, G_n(t))$ with respect to each $F(T_{n-1}, T_m)$ with $x$ held fixed, because at time $T_{n-1}$ the deltas are computed right after the commercial implementation of an optimal action at this time. Otherwise, when $t \in (T_{n-1}, T_n)$, inventory is fixed based on the prior decision at $T_{n-1}$.

By Ito’s lemma, the dynamics of $U_n(t, x, G_n(t))$ in the interval $(T_{n-1}, T_n]$ are

$$dU_n(t, x, G_n(t)) = \left[\right]dt + \sum_{m \in \mathbb{N}_n} \Delta_{n,m}(t, x, G_n(t))dF(t, T_m),$$

(8)

where

$$\left[\right] := \left[ \frac{U_n(t, x, G_n(t))}{\partial t} + \frac{1}{2} \sum_{m \in \mathbb{N}_n} \sum_{l \in \mathbb{N}_n} \frac{\partial^2 U_n(t, x, G_n(t))}{\partial F(t, T_m) \partial F(t, T_l)} \sum_{k \in K} \sigma_{m,k,n-1} F(t, T_m) \sigma_{l,k,n-1} F(t, T_l) \right].$$

As stochastic variability only arises from the last term on the right hand side of (8), this variability can be eliminated by continuously shorting the delta of each relevant futures contract. This is the bucket hedging approach.

When the number of factors driving the evolution of the futures curve is less than the number of futures contracts, that is, $K < N-n$, bucket hedging includes $N-n-K$ redundant futures trading positions. Factor hedging eliminates this redundancy by hedging the $K$ factors with just $K$ futures positions. These positions are not unique. In principle, any $K$ futures contracts can be used as long as their factor loadings are linearly independent. We include in set $\mathcal{M}_n$ the maturity labels of the $K$ futures contracts used for hedging, and collect the residual maturity labels in set $\mathcal{L}_n := \mathcal{N}_n \setminus \mathcal{M}_n$. 

10
Denote by \( q_{n,m}(t, x, G_n(t)) \) the replicating position corresponding to contract \( m \in \mathcal{N}_n \) at time \( t \) when hedging \( U_n(t, x, G_n(t)) \). With factor hedging we define \( q_{n,l}(t, x, G_n(t)) := 0 \) for all \( l \in \mathcal{L}_n \) and are interested in determining the positions \( q_{n,m}(t, x, G_n(t)) \) for all \( m \in \mathcal{M}_n \). Consider a portfolio that is long physical storage and continuously shorts these positions. The dynamics of its value \( \Pi_n(t, x, G_n(t)) \) satisfy the stochastic differential equation

\[
d\Pi_n(t, x, G_n(t)) = dU_n(t, x, G_n(t)) - \sum_{m \in \mathcal{M}_n} q_{n,m}(t, x, G_n(t))dF(t, T_m).
\]  

(9)

Using \( 8 \) and \( 1 \), \( 9 \) can be rearranged as

\[
d\Pi_n(t, x, G_n(t)) = [\cdot]dt + \sum_{m \in \mathcal{N}_n} \Delta_{n,m}(t, x, G_n(t))dF(t, T_m) - \sum_{m \in \mathcal{M}_n} q_{n,m}(t, x, G_n(t))dF(t, T_m)
\]

\[
- \sum_{m \in \mathcal{M}_n} q_{n,m}(t, x, G_n(t)) F(t, T_m) \sum_{k \in \mathcal{K}} \sigma_{m,k,n-1}dZ_k(t)
\]

\[
- \sum_{m \in \mathcal{M}_n} q_{n,m}(t, x, G_n(t)) F(t, T_m) \sigma_{m,k,n-1}
\]

\[
\int \left[ \sum_{m \in \mathcal{N}_n} \Delta_{n,m}(t, x, G_n(t)) F(t, T_m) \sigma_{m,k,n-1} + \sum_{m \in \mathcal{M}_n} q_{n,m}(t, x, G_n(t)) F(t, T_m) \sigma_{m,k,n-1} \right] dZ_k(t).
\]

(10)

It follows from \( 9 \) and \( 10 \) that the stochastic variability is hedged when

\[
\sum_{m \in \mathcal{M}_n} q_{n,m}(t, x, G_n(t)) F(t, T_m) \sigma_{m,k,n-1} = \sum_{m \in \mathcal{N}_n} \Delta_{n,m}(t, x, G_n(t)) F(t, T_m) \sigma_{m,k,n-1}, \forall k \in \mathcal{K},
\]

(11)

which is a system of \( K \) linear equations in the \( K \) unknowns \( q_{n,m}(t, x, G_n(t)), m \in \mathcal{M}_n \). The \( K \) trading positions can be obtained by solving for these unknowns in \( 11 \).

Proposition 2 provides insight into the structure of the factor hedging trading positions:

Each of these positions is the sum of the delta corresponding to a traded futures contract and a linear combinations of the deltas corresponding to the redundant futures contracts. This proposition uses some additional notation. Given \( n \in \mathcal{N} \) such that \( N - n > K \), let \( A_{n-1} \) and \( B_{n-1} \) denote the \((K \times K)\) and \((N-n-K) \times K)\) matrices of factor loading coefficients in model \( 1 \) for maturity labels in sets \( \mathcal{M}_n \) and \( \mathcal{L}_n \), respectively, as of time \( T_{n-1} \). The inverse of \( A_{n-1} \) is \( \zeta_{n-1}/|A_{n-1}| \), where \( \zeta_{n-1} \) is the transpose of the matrix of cofactors of \( A_{n-1} \) (if \( K = 1 \) then the cofactor matrix is just the scalar 1) and \(|A_{n-1}|\) is the determinant of \( A_{n-1} \). The element in position \( (l - (n + K) + 1, m - n + 1) \) of matrix \( B_{n-1} \zeta_{n-1} \) is \( \gamma_{l,m,n-1} \), with \( l \in \mathcal{L}_n \) and \( m \in \mathcal{M}_n \).
**Proposition 2** (Factor hedging positions). Pick \( n \in \mathbb{N} \) such that \( N - n > K \) and let \( t \in [T_{n-1}, T_n) \). Suppose that the determinant \( |A_{n-1}| \) is different from zero. For all traded maturities \( m \in \mathcal{M}_n \), it holds that

\[
q_{n,m}(t, x, G_n(t)) = \Delta_{n,m}(t, x, G_n(t)) + \sum_{l \in \mathcal{L}_n} \Delta_{n,l}(t, x, G_n(t)) \frac{F(t, T_l)}{F(t, T_m)} \frac{\gamma_{l,m,n-1}}{|A_{n-1}|} \quad (12)
\]

The arguments used so far in discussing factor hedging can be generalized when the hedging portfolio includes \( K + 1, \ldots, \) or \( N - 2 \) contracts. In other words, bucket and factor hedging are two particular cases of a family of hedging approaches based on constructing equivalent hedging portfolios that include \( K, K+1, \ldots, N-1 \) contracts; factor hedging with \( K \) contracts and bucket hedging with \( N-1 \) contracts. Example 1 in Online Appendix B illustrates bucket and factor hedging along with an equivalent hedging approach.

**5.2 Computing Deltas**

The storage deltas are critical to both bucket and factor hedging. Proposition 3, based on Lemma 1, establishes basic properties of the optimal value function and basestock targets. These properties are used to establish Proposition 4, which provides a pathwise expression for the deltas. All of these results are based on Assumption 1. It essentially states that \( C^I, C^W, \) and \( \bar{x} \) are rational numbers. This assumption plays a critical role in establishing part (b) of Proposition 3.

**Assumption 1** (Capacities and maximum space). The capacity limits \( C^I \) and \( C^W \) as well as the maximal inventory level \( \bar{x} \) are integer multiples of some positive real number.

We denote by \( Q \) the largest common factor of \( C^I, C^W, \) and \( \bar{x} \). Lemma 1 is related to Propositions 2 and 3 in Secomandi [41].

**Lemma 1** (Characterization). Under Assumption 1, in every stage \( n \in \mathbb{N} \):

(a) The function \( V_n(x, F_n) \) is piecewise linear continuous in inventory \( x \in \mathcal{X} \) with break points in set \( Q := \{0, Q, 2Q, \ldots, \bar{x}\} \) for each given futures curve \( F_n \in \mathbb{R}^{N-n}_+ \);

(b) The basestock levels \( b_n(F_n) \) and \( \bar{b}_n(F_n) \) can be taken to be in set \( Q, \forall F_n \in \mathbb{R}^{N-n}_+ \).

We define the finite set of feasible actions at inventory level \( x \in \mathcal{X} \) as

\[
\mathcal{A}'(x) := \{(x - \bar{x}) \vee C^I, \ldots, x - (\lfloor x/Q \rfloor + 2)Q, x - (\lfloor x/Q \rfloor + 1)Q \} \cup \{0\} \\
\cup \{x - \lfloor x/Q \rfloor Q, x - (\lfloor x/Q \rfloor - 1)Q, \ldots, x \wedge C^W\},
\]

with \( \{0\} \) removed if it otherwise were a duplicate.
Proposition 3 (Representation and Lipschitz continuity). Under Assumption 1, in every stage $n \in \mathcal{N}$:

(a) For all inventory levels $x \in \mathcal{X}$ and futures curves $F_n \in \mathbb{R}_{+}^{N-n}$ it holds that

$$V_n(x, F_n) = \max_{a \in A'(x)} p(a, s_n) + \delta E \left[ V_{n+1}(x_n - a, F_{n+1}) \mid F'_n \right];$$

(b) For each given inventory $x \in \mathcal{X}$ it holds that the function $V_n(x, F_n)$ is Lipschitz continuous in the futures curve $F_n \in \mathbb{R}_{+}^{N-n}$; that is, there exists $L_n(x) \in \mathbb{R}_{+}$ such that

$$\left| V_n(x, F^2_n) - V_n(x, F^1_n) \right| \leq L_n(x) \sum_{m=n}^{N} |F^2_{n,m} - F^1_{n,m}|, \forall F^1_n, F^2_n \in \mathbb{R}_{+}^{N-n}.$$

In Proposition 4 we indicate by $x^*_m$ the random variable that denotes the optimal inventory level in stage $m \in \mathcal{N} \setminus \{0\}$. We also let $1\{\cdot\}$ be the indicator function of event $\cdot$.

Proposition 4 (Pathwise deltas). Under Assumption 1, for every $n \in \mathcal{N} \setminus \{0\}$ it holds that

$$\Delta_{n,m}(t, x_n, G_n(t)) = \frac{\mathfrak{F}(t, T_m)}{F(t, T_m)} E[(\phi I\{x^*_m \in [0, b_m(F_m))] + \phi W 1\{x^*_m \in (b_m(F_m), x_n)\}) s_m a_m(x^*_m, F_m) \mid x_n, G_n(t)],$$

for all $m \in \mathcal{N}_n$, $x_n \in \mathcal{X}$, $t \in [T_{n-1}, T_n)$, and $G_n(t) \in \mathbb{R}_{+}^{N-n}$.

If an optimal inventory trading policy were easily computable, expression (13) could be used to estimate its associated deltas by Monte Carlo simulation. In §6, we use this expression in a heuristic fashion to estimate the deltas associated with the rolling intrinsic policy. We address the question of the accuracy of these estimates in §6.1.

6. Model Error Analysis

Before discussing our model error results, we present the valuation and hedging methods that we use, and our estimation of the price model parameters on NYMEX data.

6.1 Heuristic Valuation and Hedging

Although with very few factors one could reformulate model (3)-(4) to feature a reduced price state space, discretize the evolution of the relevant price state variables, and solve the resulting model to optimality, this would quickly become intractable with high dimensional price models. Thus, we value storage heuristically using the rolling intrinsic policy, which is widely used among practitioners (Maragos [34], Gray and Khandelwal [28, 27]) and has been shown to be nearly optimal (Lai et al. [32]).
Specifically, we obtain this policy by periodically reoptimizing the intrinsic value model, which is the deterministic equivalent of model (3)-(4), within a Monte Carlo simulation of the evolution of the futures curve. We denote by $V^I_n(x_n; F_0)$ the optimal value function of the intrinsic value model in stage $n$ with inventory level $x_n$ when this model is solved at time 0 using the futures curve available at this time, $F_0$. In stage 0, the intrinsic value model computes an optimal inventory policy that only considers the information available at time $T_0$ as follows:

$$V^I_N(x_N; F_0) := 0, \quad \forall x_N \in \mathcal{X},$$

$$V^I_n(x_n; F_0) := \max_{a \in \mathcal{A}(x_n)} p(a, F_{0,n}) + \delta V^I_{n+1}(x_n - a; F_0), \quad \forall n \in \mathcal{N}, \forall x_n \in \mathcal{X}. \quad (15)$$

The rolling intrinsic policy implements the optimal stage 0 action of this model and computes updated optimal actions for each of the remaining stages by sequentially simulating futures curve realizations, reoptimizing the resulting intrinsic model given the updated inventory and price information, and implementing the decision associated with the first stage of each such reoptimized intrinsic model. We estimate the time $T_0$ value of the rolling intrinsic policy as the average total discounted cash flows obtained by simulating this strategy over 10,000 simulated futures curve paths. Excluding the effects of simulation error, the resulting estimate is a lower bound on the value of storage.

For hedging, we apply in a heuristic fashion the pathwise deltas from Proposition 4 to compute the deltas associated with the rolling intrinsic policy. That is, we replace the optimal action in (13) with the action obtained by the rolling intrinsic policy and average the values of the resulting estimator computed by Monte Carlo simulation. We implement factor hedging as explained in §5 by using the estimates of the deltas so computed.

Our delta estimation approach may yield biased estimates, but it can be implemented efficiently, that is, simultaneously with the valuation of storage. Alternatively, we could compute deltas by resimulation, that is, by perturbing upward and downward the price of the relevant futures contract, while keeping all the other contract prices unchanged, computing the storage values for each of the two resulting futures curves, and estimating the corresponding delta as a central difference (Glasserman [26, §7.1]). Resimulation is both biased and computationally expensive (Glasserman [26, §7.1]). The main difficulty with resimulation in the storage context is its high computational requirement, which makes it infeasible for our purposes. Our computational results suggest that our approach to estimating the deltas is effective.
6.2 Data and Price Model Estimation

We use ten years of NYMEX natural gas futures price data from January 1997 to December 2006 to estimate the loading coefficients of the multi-factor price model (1). Figure 1 displays the daily evolution of the first twenty-three NYMEX natural gas futures prices over one representative year.

We use the estimation approach of Blanco et al. [5] to calibrate our futures price model. On each trading day, we compute the daily changes of the futures prices corresponding to the first twenty-three maturities. We do this because we analyze storage contracts with twenty-four month terms, as in Lai et al. [32], and value these contracts at the start of their terms. We segregate these price changes into twelve subsets, each including the daily price changes for all the relevant maturities during a specific month, over the ten year sample. We perform twelve separate PCAs on each of these monthly subsets to obtain twelve sets of monthly factor loading coefficients for price model (1). Thus, for a given maturity we have twelve sets of loading factors, one for each trading month.

Figure 2 displays the percentages of the total variance in the observed price changes explained by the first four factors for each of the twelve PCAs. The first factor captures roughly
Figure 2: The percentage of the total variance explained by the first four factors in a monthly PCA analysis over January 1997 to December 2006.

86.0-93.1% of this variance, the second factor 5.2-9.6%, the third factor 0.6-3.4%, and the fourth factor always less than 1%. Thus, a four factor model explains more than 99% of the total variance in the observed price changes.

Figure 3 displays the loading coefficients of all the relevant maturities on the first four factors, for each of the twelve estimated models. For expositional convenience we distinguish between months in the heating season (November-March), panels (a), (c), (e), and (g), and those outside this period, panels (b), (d), (f), and (h). Typically, the coefficients decrease rapidly, with magnitudes of the loading coefficients of a given factor typically less than half of those corresponding to the previous factor.

The first factor changes the slope of the natural gas futures term structure. In particular, the first factor induces positively correlated shocks along the futures term structure (since the coefficients are of the same sign), but it moves the short end of the term structure more than the long end (since the coefficients decline in magnitude in time to delivery). The second factor also shocks the slope of the futures term structure, but the changes at the long and short ends of the curve are negatively correlated (since the coefficients change sign). In contrast, the third factor shocks the curvature of the futures term structure (since the signs of short-term and long-
Figure 3: The estimated loading coefficients of the first four factors for maturities one to twenty-three for the price model (1) in each relevant month from a monthly PCA over January 1997 to December 2006.
term futures coefficients have the opposite sign from the intermediate term coefficients). The fourth factor induces random “squiggles” along the futures term structure (since the signs of the coefficients change multiple times). In terms of seasonality, the shocks to the term structure slope induced by the first factor are greater in the heating months than in the rest of the months (since the magnitude of the short-term/long-term coefficient differences are greater). There also appear to be seasonalities in the location and shape of the curvature and squiggle changes.

Since the factor loading coefficients are the same under the risk neutral and objective futures curve dynamics, the factor loading coefficients also have implications for the risk neutral dynamics of the natural gas spot prices. In the case of the first factor, the coefficients imply that spot prices mean revert over time under the risk neutral measure (since futures prices, which are risk neutral expected spot prices, respond less and less to current shocks as the time to delivery increases). The second factor induces oscillations in which risk neutral expected spot prices initially move in one direction, but then later have a persistent move in the opposite direction. Lastly, the third and fourth factors induce different forms of largely transitory risk neutral oscillations in spot prices.

6.3 Other Parameters

We normalize the maximum storage space to 1 mmBtu. Consistent with Lai et al. [32], we set the injection and withdrawal capacities, respectively, equal to 0.3 and 0.6 mmBtu per month, and the injection and withdrawal costs, respectively, equal to $0.02 and $0.01 per mmBtu. In our valuation analysis and our hedging analysis with simulated data we set the initial number of stages equal to twenty-four, that is \( N = 24 \). In our hedging analysis with real data, we set the initial number of stages equal to twelve, that is, \( N = 12 \), to increase the number of usable price samples. For simplicity, we use a constant risk-free interest rate equal to 5.05%.

6.4 Valuation Results

Figure 4 displays the value of storage as a function of the number of factors on four valuation dates: 3/1/2006, 6/1/2006, 8/31/2006, and 12/1/2006 (as in Lai et al. [32]), representing different seasons of the year. Spring and Winter are part of the heating season. Summer and Fall are not. The displayed values of storage vary in the ranges $4.333-4.595, $5.009-5.169, $4.310-4.519, and $1.217-1.536 in Spring, Summer, Fall, and Winter, respectively. The Monte Carlo standard errors of these values range from $0.012 to $0.018.

In all seasons, adding factors to the price model initially increases the value of storage
somewhat, but then the valuation flattens out. The initial increases are about 4-5% of the maximal valuation in the Spring, Summer, and Fall instances, and roughly 20% of this valuation in the Winter instance. In Spring, Summer, and Fall three factors are sufficient to achieve valuations that are 98% of the maximum; in Winter, this requires six factors. This is consistent with the earlier observation that four factors explain more than 99% of the observed total daily price change variance. Thus, parsimonious price models can reproduce essentially the same valuations as higher, and in particular full dimensional price models. Our finding suggests that model error due to omitted factors in an empirically calibrated version of model (1) is not problematic for storage valuation, once the most important three to six statistical factors are included.

6.5 Hedging Results with Simulated Data

As natural gas storage is not marked-to-market, we cannot employ market prices for storage to measure hedging effectiveness. Thus, in this subsection we focus on how effective futures hedging is in replicating the cash flows generated by physical trading under the rolling intrinsic policy along a simulated futures curve sample path.

Given a futures curve at time 0, we define by $\omega$ a sample path of $N$ futures curve realizations
at times 0 through $T_{N-1}$ from the price model (1) with a given true number of factors. We take the NYMEX closing futures prices on 6/1/2006 as the time 0 futures curve. For a given assumed number of factors, we define the cumulative accrued value of the cash flows of the rolling intrinsic policy from time 0 through time $T_{N-1}$ along path $\omega$ as

$$P(\omega) := \sum_{n \in \mathcal{N}} \delta^{N-1-n} p(a_{n+1}^{RI}(x_n, F_n), s_n; \omega),$$

where $a_{n}^{RI}(x_n, F_n)$ is the action taken by the rolling intrinsic policy in state $(x_n, F_n)$ in stage $n$ ($P(\omega)$ is not superscripted by RI to ease the notation). We denote the time $T_{n-1}$ vector of replicating positions corresponding to a given hedging method as $q_{n}(T_{n-1}, \cdot, F_{n-1}) := (q_{n,m}(T_{n-1}, \cdot, F_{n-1}), m \in \mathcal{N}_{n})$. We define the cash flow generated at time $T_{n}$ along path $\omega$ by taking positions $q_{n}(T_{n-1}, \cdot, F_{n-1})$ at time $T_{n-1}$ and holding them until time $T_{n}$ as

$$\psi_{n}(q_{n}(T_{n-1}, \cdot, F_{n-1}), F'_{n-1}, F_{n}; \omega) := \sum_{m \in \mathcal{N}_{n}} q_{n,m}(T_{n-1}, \cdot, F'_{n-1}) (F_{n,m} - F_{n-1,m}) (\omega).$$

We define the cumulative accrued value of these cash flows from time $T_{1}$ through time $T_{N-1}$ as

$$\Psi(\omega) := \sum_{n \in \mathcal{N}\setminus\{0\}} \delta^{N-1-n} \psi_{n}(q_{n-1}(T_{n-1}, x_{n-1} - a_{n-1}^{RI}(x_{n-1}, F_{n-1}), F'_{n-1}), F_{n-1}, F_{n}; \omega).$$

These definitions assume monthly rebalancing of the replicating positions. We could adapt our model to deal with more frequent rebalancing by increasing the granularity of our futures curve simulations.

We denote by $\text{VAR}(P)$ and $\text{VAR}(P - \Psi)$ the cross-path variances of the total physical cash flows $P$ and residual total cash flows after hedging $P - \Psi$, respectively. Our metric of hedging effectiveness is the reduction in $\text{VAR}(P)$ achieved by hedging:

$$100 \left[ 1 - \frac{\text{VAR}(P - \Psi)}{\text{VAR}(P)} \right]. \quad (16)$$

The hedging effectiveness of a perfect hedge is 100%. For each valuation model and hedging method pair that we consider, we use Monte Carlo simulation to estimate $\text{VAR}(P)$ and $\text{VAR}(P - \Psi)$, and hence (16).

Our metric of hedging effectiveness is related to that used by Driessen et al. [19], who measure the ratio of the sample variances of biweekly hedging errors and changes in the market price of a particular derivative investment. However, we work with total physical and hedging cash flows rather than biweekly hedging errors and market price changes, as natural gas storage is not marked-to-market.
Table 1: Hedging effectiveness ratios of naïve factor hedging on simulated data. The emphasized values correspond to the case of no model error.

<table>
<thead>
<tr>
<th>True Number of Factors</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
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<tbody>
<tr>
<td>1</td>
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<td>96.02</td>
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<td>84.00</td>
<td>99.62</td>
<td>99.61</td>
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</table>

We consider price models with the following number of true and assumed factors: 1, 2, 3, 5, 10, 15, 20 and 23. We rebalance the financial trading positions once every month. To manage the computational effort, we use a Monte Carlo sample size of thirty paths. Obtaining the relevant values associated with one sample path for a given triple of true number of factors, assumed number of factors, and hedging method requires between 6.9 and 7.7 Cpu minutes, depending on the number of true and assumed factors, using the `g++ version 4.3.0 20080428 (Red Hat 4.3.0-8)` compiler on a 64 bits `Monarch Empre 4-Way Tower Server` with four `AMD Opteron 852 2.6GHz` processors, each with eight `DDR-400 SDRAM of 2 GB` running `Linux Fedora 11`; the stated Cpu times correspond to using a single processor. To control for sampling error, we use common random numbers when generating price samples across different models that share common factors.

Our hedging analysis considers three cases: No model error, model error due to omitted factors, and model error due to excess assumed factors.

For a given assumed number of factors, factor hedging involves choosing a subset of futures contracts at each date $T_{n-1}$ and inverting the square submatrix $A_{n-1}$ of their loading coefficients to compute the trading positions (Proposition 2). Call $A_{n-1}$ the position generating matrix at time $T_{n-1}$. To gain some insight into how this choice affects the performance of factor hedging, we first run some preliminaries experiments with a naïve version of factor hedging that takes positions in the shortest maturity contracts. Table 1 reports these results. Without model error, naïve factor hedging performs well, as expected, with hedging effectiveness ratios above 96.19%. In contrast, with the notable exceptions of one, fifteen, and twenty assumed factors, naïve factor hedging performs disastrously with model error arising from omitting factors. In this case, the hedging effectiveness ratios are large and negative; that is, hedging dramatically
Table 2: Smallest and largest hedging positions taken by naïve factor hedging across all sample paths, trading dates, and contracts traded on simulated data.

<table>
<thead>
<tr>
<th>True Number of Factors</th>
<th>Smallest Position</th>
<th>Assumed Number of Factors</th>
<th>Largest Position</th>
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</thead>
<tbody>
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<td></td>
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increases the residual cash flow variance relative to the unhedged cash flow variance. In contrast, with model error from excess assumed factors, naïve factor hedging performs as if there were no model error.

The disastrous performance of naïve factor hedging with omitted factors and between two and ten assumed factors is due to this method exhibiting extremely large trading positions (in absolute value) that interact with model error. Table 2 displays the smallest and largest hedging positions taken by naïve factor hedging across all sample paths, trading dates, and contracts traded. These large positions are induced by the small determinants of the matrices used to generate them, as the determinant of this matrix appears in the denominator of expression (12). Without model error, these positions, even if large, replicate well the value of storage because they are consistent with the realized price changes, as they arise from the assumed model (however they would be difficult to trade in practice). With omitted factors, these positions are structurally wrong, as they are inconsistent with the realized price changes, and their magnitudes (in absolute value) magnify the model error embedded in the realized price changes. This generates severe discrepancies between the realized physical trading cash flows and the replicated cash flows.

The exceptions to the poor performance in Table 1 deserve comment. With a single assumed factor, the stated matrix inversion reduces to a division by a scalar and does not generate large trading positions. With fifteen assumed factors the trading positions can be large but the empirically calibrated magnitude of the model error is extremely small. With twenty assumed factors, the stated matrix inversion does not appear to be problematic, as factor hedging converges to bucket hedging when the number of assumed factors approaches the number of maturities in the term of the storage contract and, as discussed below, bucket hedging is remarkably robust to model error.

With excess assumed factors, factor hedging is essentially not exposed to model error. The valuations obtained when varying the number of factors are close to each other (see Figure 4), and the hedging cash flows obtained under the assumed factor model would be identical to those that would be obtained under the true model by trading the same contracts, albeit possibly in different amounts due to possible differences in the computed deltas (see Example 2 in Online Appendix B), which, however, turn out to be very small (this is discussed below).

The poor performance of naïve factor hedging with omitted factors motivates us to develop an enhanced implementation of factor hedging based on using position generating matrices with large determinants. For a given number of factors, in each stage we examine all the possible
Table 3: Hedging effectiveness ratios of enhanced factor hedging on simulated data. The emphasized values correspond to the case of no model error.

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<th>True Number of Factors</th>
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<tr>
<td>10</td>
<td>79.44</td>
<td>94.78</td>
<td>96.37</td>
<td>98.79</td>
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<tr>
<td>15</td>
<td>81.01</td>
<td>94.75</td>
<td>96.45</td>
<td>98.81</td>
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<td>99.64</td>
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<tr>
<td>20</td>
<td>80.95</td>
<td>94.80</td>
<td>96.38</td>
<td>98.75</td>
<td>99.52</td>
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<td>23</td>
<td>79.90</td>
<td>94.61</td>
<td>96.18</td>
<td>98.74</td>
<td>99.49</td>
<td>99.60</td>
<td>99.62</td>
<td>99.62</td>
<td>99.61</td>
</tr>
</tbody>
</table>

Position generating matrices and select the one with the largest determinant. We do this before the trading simulation.

Table 3 displays the hedging effectiveness of enhanced factor hedging. Without model error or with model error and excess factors, enhanced factor hedging performs extremely well, as expected, with hedging effectiveness ratios above 99.60%. With omitted factors, the performance of enhanced factor hedging has improved dramatically relative to naïve factor hedging. Specifically, the hedging effectiveness ratios are 94.60% or higher in all cases with two or more assumed factors. Moreover, for a given number of true factors, the relevant ratios increase in the number of assumed factors, becoming essentially equal to one when the assumed number of factors meets or exceeds the true number of factors. The increase in these ratios is due to the obvious reduction in model error as the number of factors in the assumed model increases up to the true number of factors. For a given number of assumed factors, the relevant ratios generally decrease when the true number of factors increases and model error is caused by omitted factors. This is due to the increase in model error in these cases. The improvement in performance of enhanced factor hedging relative to naïve factor hedging is due to the smaller (absolute) values of the hedging positions of enhanced factor hedging, displayed in Tables 4. This improvement is noteworthy and underlies the importance of carefully selecting a position generation matrix when implementing factor hedging.

Having demonstrated that enhanced factor hedging has reasonably good performance, we next discuss the performance of bucket hedging. The results are in Table 5. The smallest and largest positions of bucket hedging are $-0.30$ and $0.59$ irrespective of the number of true and assumed factors. With model error and omitted factors, bucket hedging outperforms enhanced factor hedging, with all the relevant ratios above 99.60%, irrespective of the number of assumed
Table 4: Smallest and largest hedging positions taken by enhanced factor hedging across all sample paths, trading dates, and contracts traded on simulated data.

<table>
<thead>
<tr>
<th>True Number</th>
<th>Assumed Number of Factors</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>23</th>
</tr>
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<td>of Factors</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td>-0.26</td>
<td>-0.72</td>
<td>-0.72</td>
<td>-0.92</td>
<td>-0.65</td>
<td>-0.71</td>
<td>-0.40</td>
<td>-0.30</td>
</tr>
<tr>
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<td></td>
<td>-0.25</td>
<td>-0.75</td>
<td>-0.77</td>
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<td>-0.68</td>
<td>-0.71</td>
<td>-0.42</td>
<td>-0.30</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>-0.25</td>
<td>-0.75</td>
<td>-0.80</td>
<td>-1.05</td>
<td>-0.67</td>
<td>-0.72</td>
<td>-0.42</td>
<td>-0.30</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>-0.26</td>
<td>-0.75</td>
<td>-0.80</td>
<td>-1.07</td>
<td>-0.66</td>
<td>-0.74</td>
<td>-0.42</td>
<td>-0.30</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>-0.30</td>
<td>-0.74</td>
<td>-0.81</td>
<td>-1.06</td>
<td>-0.67</td>
<td>-0.75</td>
<td>-0.41</td>
<td>-0.30</td>
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<tr>
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<td>-0.74</td>
<td>-0.80</td>
<td>-1.06</td>
<td>-0.66</td>
<td>-0.75</td>
<td>-0.41</td>
<td>-0.30</td>
</tr>
<tr>
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<td>-0.74</td>
<td>-0.81</td>
<td>-1.06</td>
<td>-0.66</td>
<td>-0.75</td>
<td>-0.41</td>
<td>-0.30</td>
</tr>
<tr>
<td>23</td>
<td></td>
<td>-0.30</td>
<td>-0.74</td>
<td>-0.81</td>
<td>-1.07</td>
<td>-0.66</td>
<td>-0.76</td>
<td>-0.41</td>
<td>-0.30</td>
</tr>
</tbody>
</table>

Table 5: Hedging effectiveness ratios of bucket hedging on simulated data. The emphasized values correspond to the case of no model error.

<table>
<thead>
<tr>
<th>True Number</th>
<th>Assumed Number of Factors</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>10</th>
<th>15</th>
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<td>99.73</td>
<td>99.73</td>
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<td>99.82</td>
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<tr>
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<td>99.63</td>
<td>99.61</td>
<td>99.60</td>
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<tr>
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<td></td>
<td>99.61</td>
<td>99.71</td>
<td>99.68</td>
<td>99.65</td>
<td>99.64</td>
<td>99.64</td>
<td>99.64</td>
<td>99.64</td>
</tr>
</tbody>
</table>
factors. Without model error or with model error and excess factors, the performances of bucket hedging and enhanced factor hedging are equivalent. The very similar performance of bucket hedging across different cases is due to the deltas computed in each of these cases being essentially the same to those computed without model error. This is remarkable, especially as bucket hedging requires no optimization.

Note that all of the bucket hedges use all the relevant futures contracts, e.g., twenty-three in stage 0. The assumed number of factors refers to the number of factors driving the futures curve dynamics in the price model used for storage valuation. In practice, traders base bucket hedges on full dimensional price models, but bucket hedging and the number of factors in the price model are separable. Our experiments show that bucket hedging based on a lower dimensional price model gives similar hedging results.

6.6 Hedging Results with Real Data

We conclude our investigation by comparing the hedging results obtained on real data, for which the true price model is unknown. We consider both enhanced factor and bucket hedging, varying the number of assumed factors. Since history contains a single “sample path,” we split the data into subperiods to obtain more than one subsample. Table 6 presents these subperiods. We perform ten in-sample and eight out-of-sample hedging experiments for a twelve month storage contract.

Table 7 reports the relevant results. With one assumed factor, the hedging effectiveness ratios of enhanced factor hedging are 94.81% and 86.65% for the in-sample and the out-of-sample cases, respectively. With two or more assumed factors, these ratios are at least 99.67% and 98.42%. The relevant ratios of bucket hedging are all 99.79% or higher. These results
confirms those obtained with simulated data when comparing these methods: Enhanced factor hedging is effective, but bucket hedging is yet more effective, even in the possible presence of model error. They also suggest that three factors should be sufficient for hedging natural gas storage contracts with twelve month terms.

6.7 Summary

The results discussed in this section show that price models with three to six factors are effective at valuing and hedging natural gas storage contracts with up to twenty-four month terms, yielding valuations and hedging performances that are comparable to those obtained with full dimensional price models, provided that enhanced factor hedging and/or bucket hedging are employed for hedging purposes. The consistently excellent performance of bucket hedging irrespective of the number of factors included in the price model is remarkable.

7. Conclusions

In this paper we study the valuation and hedging of natural gas storage contracts with model error using a family of empirically calibrated factor models of the natural gas futures curve evolution. Model error is arguably important in practice when using reduced form price models, such as the one we consider.

In terms of valuation, the impact of model error is modest provided that the most important three to six statistical factors are included in the futures price model. This contrasts with the apparent widespread use among practitioners of full dimensional price models, e.g., using a twenty-four factor model to value a twenty-four month storage contract.
In terms of hedging, not all delta hedging strategies are equivalent in the presence of model error. In particular, the impact of model error, in particular on the performance of naïve factor hedging, can be dramatically reduced by employing an enhanced version of factor hedging that carefully selects which contracts to use or, alternatively, by using bucket hedging. Practitioners are likely to favor bucket hedging over enhanced factor hedging, due to the simplicity of bucket hedging and its remarkable robustness to model error irrespective of the number of factors used for valuation. In practice, bucket hedging is sometimes implemented based on a full dimensional price model. We show that bucket hedging does not require the use of a full dimensional price model for accurate valuation. This suggests there may be redundancy in the number of factors used for valuation, but not in the number of contracts used for hedging, in part of current practice.

Enhanced factor hedging may offer an appealing alternative to bucket hedging when trading in all the contracts in the term of a long dated storage contract is not viable, e.g., due to reduced liquidity. This is an appealing topic for additional research.

We value natural gas storage contracts using the rolling intrinsic policy. Notwithstanding the near optimal performance of this heuristic with a specific full dimensional price model (Lai et al. [32]), a promising area for further research is the development of valuation methods that leverage our finding that lower dimensional price models can deliver similar valuations of full dimensional models. The approximate dynamic programming literature (de Farias and Van Roy [18], Bertsekas [2, Chapter 6], Adelman [1], Powell [36]) may prove useful for this purpose.

Acknowledgments

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References


agement Science, Norwegian School of Economics and Business Administration, Bergen, Norway, 2008.


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Online Appendix

A. Proofs

Proof of Proposition 2 (Factor hedging positions). Let $dZ(t)$ be a $K$-dimensional vector of independent Brownian motions. Denote by $dF^t$ the vector $(dF(t, T_0), \ldots, dF(t, T_{N-1}))$, and by $dF^{tA}(t)$ and $dF^{tB}(t)$, respectively, the vectors of its components corresponding to maturity labels in sets $\mathcal{M}_n$ and $\mathcal{L}_n$. Write model (1) in vector notation as $dF^{tA}(t)/F^{tA}(t) = A^{-1}_n dZ(t)$ and $dF^{tB}(t)/F^{tB}(t) = B^{-1}_n dZ(t)$, where "/" indicates componentwise division. Using the assumption that $|A^{-1}_{n-1}| \neq 0$, it is not hard to show that

$$dF^{tB}(t)/F^{tB}(t) = B^{-1}_n A^{-1}_{n-1} dF^{tA}(t)/F^{tA}(t),$$

which can be equivalently written as

$$dF(t, T_l) = \sum_{m \in \mathcal{M}_n} \frac{F(t, l)}{F(t, m)} \frac{\gamma_{l,m,n-1}}{|A^{-1}_{n-1}|} dF(t, T_m), \forall l \in \mathcal{L}_n. \quad (17)$$

Using (17), the second term on the right hand side of (8) can be written as

$$\sum_{m \in \mathcal{M}_n} \Delta_{n,m}(t, x, G_n(t)) dF(t, T_m) = \sum_{m \in \mathcal{M}_n} \Delta_{n,m}(t, x, G_n(t)) dF(t, T_m)$$

$$+ \sum_{l \in \mathcal{L}_n} \Delta_{n,l}(t, x, G_n(t)) dF(t, T_l)$$

$$= \sum_{m \in \mathcal{M}_n} \Delta_{n,m}(t, x, G_n(t)) dF(t, T_m)$$

$$+ \sum_{l \in \mathcal{L}_n} \Delta_{n,l}(t, x, G_n(t)) \sum_{m \in \mathcal{M}_n} \frac{F(t, l)}{F(t, m)} \frac{\gamma_{l,m,n-1}}{|A^{-1}_{n-1}|} dF(t, T_m)$$

$$= \sum_{m \in \mathcal{M}_n} \Delta_{n,m}(t, x, G_n(t)) dF(t, T_m)$$

$$+ \sum_{m \in \mathcal{M}_n} \sum_{l \in \mathcal{L}_n} \Delta_{n,l}(t, x, G_n(t)) \frac{F(t, l)}{F(t, m)} \frac{\gamma_{l,m,n-1}}{|A^{-1}_{n-1}|} dF(t, T_m)$$

$$= \sum_{m \in \mathcal{M}_n} (\cdot)_{n,m} dF(t, T_m),$$

with $(\cdot)_{n,m} := \Delta_{n,m}(t, x, G_n(t)) + \sum_{l \in \mathcal{L}_n} \Delta_{n,l}(t, x, G_n(t)) [F(t, l)/F(t, m)] (\gamma_{l,m,n-1}/|A^{-1}_{n-1}|)$. Expression (12) holds with $q_{n,m}(t, x, G_n(t)) = (\cdot)_{n,m}$, for all $m \in \mathcal{M}_n$. \hfill \Box

We introduce some additional notation used in the proofs below.

We define by $W_n(x, F'_n) := \delta E[V_{n+1}(x, F_{n+1})|F'_n]$ the discounted expected continuation value in each stage $n < N$ given the inventory level $x$ and the futures curve $F'_n$. We use this notation in the proofs of Lemma 1 and Propositions 3 and 4.
We denote by $v_n(x, a, F_n)$ the objective function of the maximization on the right hand side of (4). We use this notation in the proofs of Propositions 3 and 4 and Lemma 3.

Given $n$ and $m \in N_{n+1}$, $t \in [T_n, T_{n+1})$, and $t' \in (T_n, T_{n+1}]$ we define

$$
\beta_m(t, t', Y) := \exp \left[ \frac{1}{2} (t' - t) \sum_{k \in K} \sigma_{m,k,n}^2 + \sqrt{t' - t} \sum_{k \in K} \sigma_{m,k,n} Y_k \right]
$$

and $\beta(t, t', Y) := (\beta_m(t, t', Y), m \in N_{n+1})$. By (2), we can equivalently express $F'(t')$ given $F'(t), \beta(t, t', Y)$, where "\^\(^\cdot\)" denotes componentwise multiplication, and $F(T_{n+1})$ given $F(t)$ as $F'(t), \beta(t, T_{n+1}, Y)$. Hence, the equalities

$$
F(t, T_m) = E[F(t', T_m)|F(t, T_m)],
$$

$$
E[F(t', T_m)|F(t, T_m)] = F(t, T_m)E[\beta_m(t, t', Y)]
$$

imply that

$$
E[\beta_m(t, t', Y)] = 1. \tag{18}
$$

Finally, given $n < N - 1$, when $t = T_n$ and $t' = T_{n+1}$, we abbreviate $\beta_m(T_n, T_{n+1}, Y)$ to $\beta_m^{n+1}(Y)$ for all $m = n + 1, \ldots, N - 1$, and $\beta(T_n, T_{n+1}, Y)$ to $\beta^{n+1}(Y)$. We use this notation in the proofs of Proposition 3 and Lemma 2.

**Proof of Lemma 1 (Characterization).** The claimed properties are true in stage $N - 1$, as $V_{N-1}(x_{N-1}, s_{N-1}) = (\phi^W s_{N-1} - c^W) \uparrow (x_{N-1} \land C^W)$, and $\bar{b}_{N-1}(s_{N-1}) = 0$ and $\bar{b}_{N-1}(s_{N-1}) = \bar{\mathbb{1}} \{ \phi^W s_{N-1} - c^W \leq 0 \} (1\{ \cdot \} \text{ is the indicator function of event } \cdot)$. Make the induction hypothesis that these properties also hold in stages $n + 1, \ldots, N - 2$. Consider stage $n$. By definition, $W_n(x, F_n')$ is the discounted expected value of piecewise linear continuous functions, each with possible break points only in set $Q$, which implies that this function satisfies the same property. The quantities $b_n(F_n)$ and $\bar{b}_n(F_n)$ are optimal solutions to the following maximizations, respectively (see the proof of Theorem 1 in Secomandi [49]):

$$
\max_{x_{n+1} \in \mathcal{X}} W_n(x_{n+1}, F_n') - (\phi^I s_n + c^I) x_{n+1},
$$

$$
\max_{x_{n+1} \in \mathcal{X}} W_n(x_{n+1}, F_n') - (\phi^W s_n - c^W) x_{n+1}.
$$

Thus, $b_n(F_n)$ and $\bar{b}_n(F_n)$ can be taken to belong to set $Q$. It follows that (recalling that $C^I < 0$)

$$
V_n(x_n, F_n) = \begin{cases} 
(\phi^I s_n + c^I) C^I + W_n(x_n - C^I, F_n'), & x_n \in [0, b_0(F_n) + C^I), \\
(\phi^I s_n + c^I) \lfloor x_n - b_n(F_n) \rfloor + W_n(b_n(F_n), F_n'), & x_n \in [b_n(F_n) + C^I, b_0(F_n)), \\
W_n(x_n, F_n'), & x_n \in [b_n(F_n), \bar{b}_n(F_n)], \\
(\phi^W s_{N-1} - c^W) [x_n - \bar{b}_n(F_n)] + W_n(\bar{b}_n(F_n), F_n'), & x_n \in (\bar{b}_n(F_n) + C^W, \bar{b}_n(F_n) + C^W), \\
(\phi^W s_{N-1} - c^W) C^W + W_n(x_n - C^W, F_n'), & x_n \in (\bar{b}_n(F_n) + C^W, \bar{\mathbb{1}}].
\end{cases}
$$

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It is easy to check that this function is piecewise linear and continuous in \( x_n \) with break points in set \( Q \) for each given \( F_n \). Therefore, the claimed properties are true in stage \( n \). By the principle of mathematical induction, they hold in every stage. □

**Proof of Proposition 3 (Representation and Lipschitz continuity).** (a) This part follows directly from part (b) of Lemma 1.

(b) The claimed property holds trivially in stage \( N \) with \( L_N(x) = 0 \) for all \( x \in \mathcal{X} \). Make the induction hypothesis that this property holds in stages \( n + 1, \ldots, N - 1 \). Consider stage \( n \) and pick \( F_n^1, F_n^2 \in \mathbb{R}_{n}^{N - n} \).

Define \( C := |C^j| \cap C^W \) and \( C' := (\phi^j \cup \phi^W)C \). For each \( a \in [-C, 0) \), it holds that

\[
|p(a, s_n^2) - p(a, s_n^1)| = |(\phi^j s_n^2 + c^j)a - (\phi^j s_n^1 + c^j)a| \\
\leq \phi^j |a||s_n^2 - s_n^1| \leq \phi^j C|s_n^2 - s_n^1| \leq C'|s_n^2 - s_n^1|.
\]

For \( a = 0 \), it holds that

\[
|p(a, s_n^2) - p(a, s_n^1)| = 0 \leq \phi^W C|s_n^2 - s_n^1| \leq C'|s_n^2 - s_n^1|.
\]

For each \( a \in (0, C] \), it holds that

\[
|p(a, s_n^2) - p(a, s_n^1)| = |(\phi^W s_n^2 - c^W)a - (\phi^W s_n^1 - c^W)a| \\
\leq \phi^W |a||s_n^2 - s_n^1| \leq \phi^W C|s_n^2 - s_n^1| \leq C'|s_n^2 - s_n^1|.
\]

Thus, given \( x \in \mathcal{X} \) and \( a \in \mathcal{A}(x) \subseteq [-C, C] \), we have

\[
|p(a, s_n^2) - p(a, s_n^1)| \leq C'|s_n^2 - s_n^1|.
\]

Replacing \( W_n(x_{n+1}, F_n^{2'}) - W_n(x_{n+1}, F_n^{1'}) \) with \(|\cdot|\) for expositional convenience, it holds that

\[
|\cdot| = \delta \mathbb{E} \left[ V_{n+1}(x_{n+1}, F_n^2, \beta^{n+1}(Y)) - V_{n+1}(x_{n+1}, F_n^1, \beta^{n+1}(Y)) \right] \\
\leq \delta \mathbb{E} \left[ \sum_{m=n+1}^{N} |F_{n,m}^2 - F_{n,m}^1| \beta_m^{n+1}(Y) \right] \\
\leq \delta L_{n+1}(x_{n+1}) \sum_{m=n+1}^{N} |F_{n,m}^2 - F_{n,m}^1| \mathbb{E} \left[ \beta_m^{n+1}(Y) \right] \\
= \delta L_{n+1}(x_{n+1}) \sum_{m=n+1}^{N} |F_{n,m}^2 - F_{n,m}^1|,
\]

where the first inequality holds because \(|\mathbb{E}[\cdot]| \leq \mathbb{E}[|\cdot|]\), the second inequality follows from the induction hypothesis, and the last equality follows from (18).
Given \( x \in \mathcal{X} \) and \( a \in \mathcal{A}(x) \), inequalities (19) and (20) imply that for all \( F^1_n, F^2_n \in \mathbb{R}^{N-n}_+ \) it holds that
\[
|v_n(x, a, F^2_n) - v_n(x, a, F^1_n)| = \left| p(a, s^2_n) - p(a, s^1_n) + W_n(x_n - a, F^2_n) - W_n(x_n - a, F^1_n) \right|
\leq |p(a, s^2_n) - p(a, s^1_n)| + \left| W_n(x_n - a, F^2_n) - W_n(x_n - a, F^1_n) \right|
\leq C'|s^2_n - s^1_n| + \delta L_{n+1}(x_n - a) \sum_{m=n+1}^N |F^2_{n,m} - F^1_{n,m}|
\leq \{C' \lor [\delta L_{n+1}(x_n - a)]\} \sum_{m=n}^N |F^2_{n,m} - F^1_{n,m}|.
\]

Thus, the function \( v_n(x, a, F_n) \) is Lipschitz continuous in \( F_n \in \mathbb{R}^{N-n}_+ \) for each given \( x \in \mathcal{X} \) and \( a \in \mathcal{A}(x) \).

Part (a) of this proposition and Proposition 11.2.2(a) in Dudley [48, p. 391] imply that the claimed property holds in stage \( n \). It follows from the principle of mathematical induction that the claimed property holds in every stage. \( \square \)

Lemmas 2 and 3 are needed in the proof of Proposition 4.

**Lemma 2** (Pathwise derivatives). Under Assumption 1, given \( x \in \mathcal{X} \), for every \( n \in \mathcal{N} \setminus \{N-1\} \) and \( m \in \mathcal{N}_{n+1} \) it holds that
\[
\frac{dV_{n+1}(x, F_{n+1})}{dF_{n,m}} = \frac{\partial V_{n+1}(x, F_{n+1})}{\partial F_{n+1,m}} \frac{F_{n+1,m}}{F_n},
\]
\[
\frac{\partial E[V_{n+1}(x, F_{n+1})|F_n]}{\partial F_{n,m}} = \mathbb{E} \left[ \frac{dV_{n+1}(x, F_{n+1})}{dF_{n,m}} \bigg| F'_n \right].
\]

**Proof.** We interpret \( F_n \) as a parameter and write \( F_{n+1}(F_{n,m}) \) to explicitly indicate the dependence of \( F_{n+1} \) on this parameter. In particular, as \( F_{n+1,m} = F_{n,m}/\beta^m_{n+1}(Y) \), \( F_{n+1,m} \) depends on \( F_{n,m} \), but every other \( F_{n+1,l} \), with \( l > m \), does not.

We first show that conditions (A1)-(A4) in Appendix A of Broadie and Glasserman [47] hold with respect to \( E[V_{n+1}(x, F_{n+1}(F_{n,m}))|F_n'] \).

(A1) Under model (2) the quantity \( \partial F_{n+1,l}(F_{n,m})/\partial F_{n,m} \) exists with probability 1 because it is equal to \( \beta^m_{n+1}(Y) \) if \( l = m \) and to 0 when \( l > m \).

(A2) Fix \( x \) and let \( D^V_{n+1} \) denote the set of futures curves \( F_{n+1} \) at which \( V_{n+1}(x, F_{n+1}) \) is differentiable with respect to each element of \( F_{n+1} \). Part (b) of Proposition 3 and Rademacher’s theorem imply that \( V_{n+1}(x, F_{n+1}) \) is differentiable almost everywhere on \( \mathbb{R}^{N-n-1}_+ \) with respect to each element of \( F_{n+1} \). Price model (2) implies that \( \Pr(F_{n+1}(F_{n,m}) \in D^V_{n+1}) = 1 \) for all \( F_{n,m} \in \mathbb{R}_+ \).
(A3) This is part (b) of Proposition 3.

(A4) The random variable $F_{n+1,m}(F_{n,m})$ is almost surely Lipschitz with integrable modulus $\beta^{n,n+1}(Y)$ because $|F_{n+1,m}(F_{n,m}^2) - F_{n+1,m}(F_{n,m}^1)| = \beta^{n,n+1}(Y)|F_{n,m}^2 - F_{n,m}^1|$, for all $F_{n,m}$, $F_{n,m}^2 \in \mathbb{R}_+$, and $\mathbb{E}[\beta^{n,n+1}(Y)] = 1$. Every other random variable $F_{n+1,l}(F_{n,m})$, with $m < l$, is almost surely Lipschitz with integrable modulus 0 because each such random variable does not depend on $F_{n,m}$.

Following Broadie and Glasserman [47, p. 280], (21) holds because under conditions (A1)-(A2) the pathwise derivative $dV_{n+1}(x, F_{n+1})/dF_{n,m}$ satisfies

$$
\frac{dV_{n+1}(x, F_{n+1})}{dF_{n,m}} = \sum_{t=n+1}^{N-1} \frac{\partial V_{n+1}(x, F_{n+1})}{\partial F_{n+1,t}} \frac{\partial F_{n+1,t}}{\partial F_{n,m}} = \frac{\partial V_{n+1}(x, F_{n+1})}{\partial F_{n+1,m}} \frac{\partial \beta^{n,n+1}(Y)}{\partial F_{n,m}} = \frac{\partial V_{n+1}(x, F_{n+1}) F_{n+1,m}}{\partial F_{n+1,m} F_{n,m}}.
$$

Expression (22) follows from Proposition 1 in Broadie and Glasserman [47]. □

**Lemma 3 (Differentiability and unique optimal action).** Under Assumption 1, for every $n \in \mathbb{N}$, if $V_n(x_n, F_n)$ is differentiable with respect to each element of $F_n$ at a given futures curve $\overline{F}_n$ for given $x_n \in \mathcal{X}$, then $a^*_n(x_n, \overline{F}_n)$ is unique and

$$
\frac{\partial V_n(x_n, F_n)}{\partial F_{n,m}} \bigg|_{F_n = \overline{F}_n} = \frac{\partial v_n(x_n, a^*_n(x_n, \overline{F}_n), F_n)}{\partial F_{n,m}} \bigg|_{F_n = \overline{F}_n}, \forall m \in \mathcal{N}_n.
$$

**Proof.** Consider stage $n$. Part (b) of Proposition 3 and Rademacher’s theorem imply that the function $V_n(x_n, F_n)$ is differentiable almost everywhere in each element of $F_n$ for each given $x_n \in \mathcal{X}$. Fix $x_n$ and pick $\overline{F}_n$ such that $V_n(x_n, F_n)$ is differentiable in each element of $F_n$ at $\overline{F}_n$. In particular, this means that $\partial V_n(x_n, F_n)/\partial s_n$ exists at $\overline{s}_n$. Each function $v_n(x_n, a, F_n)$ with $a \in \mathcal{A}(x_n)$ is linear in $s_n$. This and part (a) of Proposition 3 imply that there is a unique optimal action at $\overline{F}_n$, because otherwise $\partial V_n(x_n, F_n)/\partial s_n$ would not exist at $\overline{s}_n$. Denote this action by $a^*_n(x_n, \overline{F}_n)$. This implies that $V_n(x_n, \overline{F}_n) \equiv v_n(x_n, a^*_n(x_n, \overline{F}_n), \overline{F}_n)$ in a neighborhood of $\overline{F}_n$. The assumed differentiability of $V_n(x_n, F_n)$ in each element of $F_n$ at $\overline{F}_n$ implies that $\partial V_n(x_n, F_n)/\partial F_{n,m} = \partial v_n(x_n, a^*_n(x_n, \overline{F}_n), F_n)/\partial F_{n,m}$ at $\overline{F}_n$ for all $m \in \mathcal{N}_n$. □

**Proof of Proposition 4 (Pathwise deltas).** For simplicity of exposition, we prove the claimed result for $t = 0$ and $n = 1$. The proof for the general case follows similar steps. As $F'_0 \equiv G_1(T_0)$, in this proof we use the notation $F'_0$ in lieu of the notation $G_1(T_0)$.

Suppose $m = 1$. Formula (7) and Lemma 2 imply that

$$
\frac{\Delta_{1,1}(0, x_1, F'_0)}{\delta(0, T_1)} = \frac{1}{\delta(0, T_1)} \frac{\partial U_1(0, x_1, F'_0)}{\partial F_{0,1}} = \mathbb{E}[V_1(x_1, F_1)|F'_0] = \mathbb{E} \left[ \frac{dV_1(x_1, F_1)}{dF_{0,1}} \right] |F'_0|.
$$

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It follows from Lemma 2 that

\[
\frac{dV_1(x_1, F_1)}{dF_{0,1}} = \sum_{n=2}^{N-1} \frac{\partial V_1(x_1, F_1)}{\partial F_{1,n}} \frac{\partial F_{1,n}(F_{0,1})}{\partial F_{0,1}} = \frac{\partial V_1(x_1, F_1)}{\partial F_{1,1}} \beta_{0,1}^0(Y) = \frac{\partial V_1(x_1, F_1)}{\partial F_{1,1}} \frac{s_1}{F_{0,1}}.
\]  

(24)

Pick a vector \( \mathbf{F}_1 \) at which \( V_1(x_1, F_1) \) is differentiable. Lemma 3 implies that

\[
\frac{\partial V_1(x_1, F_1)}{\partial F_{1,n}} \bigg|_{F_1=\mathbf{F}_1} = \frac{\partial v_1(x_1, a_n^*(x_1, \mathbf{F}_1), F_1)}{\partial F_{1,n}} \bigg|_{F_1=\mathbf{F}_1}, \forall n \in \mathcal{N} \setminus \{0\}.
\]  

(25)

Consider the term on the right hand side of (25) for \( n = 1 \). It follows that

\[
\frac{\partial v_1(x_1, a_n^*(x_1, \mathbf{F}_1), F_1)}{\partial s_1} \bigg|_{F_1=\mathbf{F}_1} = \frac{\partial (a_n^*(x_1, \mathbf{F}_1), s_1)}{\partial s_1} \bigg|_{s_1=\mathbf{F}_1} + \frac{\partial W_1(x_1 - a_n^*(x_1, \mathbf{F}_1), F')}{\partial s_1} \bigg|_{F'=\mathbf{F}_1}.
\]  

(26)

Define the sets \( \mathcal{X}_n^l(F_n) := [0, \bar{b}_n(F_n)) \) and \( \mathcal{X}_n^W(F_n) := (\bar{b}_n(F_n), \mathbb{F}] \). The first term on the right hand side of (26) can be expressed as follows:

\[
\frac{\partial p(a_n^*(x_1, \mathbf{F}_1), s_1)}{\partial s_1} \bigg|_{s_1=\mathbf{F}_1} = \frac{\partial (\phi^l s_1 + c^l) a_n^*(x_1, \mathbf{F}_1)}{\partial s_1} \bigg|_{s_1=\mathbf{F}_1} 1 \{ x_1 \in \mathcal{X}_n^l(\mathbf{F}_1) \}
\]

\[
+ \frac{\partial (\phi^W s_1 - c^W a_n^*(x_1, \mathbf{F}_1))}{\partial s_1} \bigg|_{s_1=\mathbf{F}_1} 1 \{ x_1 \in \mathcal{X}_n^W(\mathbf{F}_1) \}
\]

\[
= [\phi^l 1 \{ x_1 \in \mathcal{X}_n^l(\mathbf{F}_1) \} + \phi^W 1 \{ x_1 \in \mathcal{X}_n^W(\mathbf{F}_1) \}] a_n^*(x_1, \mathbf{F}_1).
\]  

(27)

As \( F' \) does not depend on \( F_{1,1} \), the second term on the right hand side of (26) is zero:

\[
\frac{\partial W_1(x_1 - a_n^*(x_1, \mathbf{F}_1), F')}{\partial s_1} \bigg|_{F'=\mathbf{F}_1} = 0.
\]  

(28)

It follows from (27) and (28) that (25) can be expressed as

\[
\frac{\partial V_1(x_1, F_1)}{\partial F_{1,n}} \bigg|_{F_1=\mathbf{F}_1} = [\phi^l 1 \{ x_1 \in \mathcal{X}_n^l(\mathbf{F}_1) \} + \phi^W 1 \{ x_1 \in \mathcal{X}_n^W(\mathbf{F}_1) \}] a_n^*(x_1, \mathbf{F}_1).
\]  

(29)

Expressions (23), (24), and (29), imply that

\[
\Delta_{1,1}(0, x_1, F'_0) = \frac{\bar{\delta}(0, T_1)}{F_{0,1}} \mathbb{E} \left[ (\phi^l 1 \{ x_1 \in \mathcal{X}_n^l(F_1) \} + \phi^W 1 \{ x_1 \in \mathcal{X}_n^W(F_1) \}) s_1 a_n^*(x_1, F_1) \right].
\]

Thus, the claimed property holds for \( m = 1 \) (recall that \( x_1 \equiv x_1^* \)).

Suppose that \( m = 2 \). Analogous to (23), Lemma 2 implies that

\[
\frac{\Delta_{1,2}(0, x_1, F'_0)}{\bar{\delta}(0, T_1)} = \mathbb{E} \left[ \frac{dV_1(x_1, F_1)}{dF_{0,2}} \bigg|_{F_0=0} \right].
\]  

(30)

Consider the same vector \( \mathbf{F}_1 \) as in the case \( m = 1 \). Analogous to (24), it follows from Lemmas 2 and 3 that

\[
\frac{dV_1(x_1, F_1)}{dF_{0,2}} \bigg|_{F_1=\mathbf{F}_1} = \frac{\partial v_1(x_1, a_n^*(x_1, \mathbf{F}_1), F_1)}{\partial F_{1,2}} \bigg|_{F_1=\mathbf{F}_1} \frac{F_{1,2}}{F_{0,2}}.
\]  

(31)
For expositional convenience, we define
\[ \partial V(x_1, a^*_1(x_1, \bar{F}_1), F_1) \bigg|_{F_1 = \bar{F}_1} = \frac{\partial p(a^*_1(x_1, \bar{F}_1), s_1)}{\partial F_{1,2}} + \frac{\partial W_1(x_1 - a^*_1(x_1, \bar{F}_1), F'_1)}{\partial F_{1,2}} \bigg|_{F'_1 = \bar{F}_1}. \]  

(32)

It is clear that the first term on the right hand side of (32) is zero:
\[ \frac{\partial p(a^*_1(x_1, \bar{F}_1), s_1)}{\partial F_{1,2}} \bigg|_{s_1 = \pi_1} = 0. \]  

(33)

By Lemma 2, the second term on the right hand side of (32) can be written as
\[ \frac{\partial W_1(x_1 - a^*_1(x_1, \bar{F}_1), F'_1)}{\partial F_{1,2}} \bigg|_{F'_1 = \bar{F}_1} = \delta \frac{\partial E[V_2(x_1 - a^*_1(x_1, \bar{F}_1), F_2)|F'_1]}{\partial F_{1,2}} \bigg|_{F'_1 = \bar{F}_1} \]
\[ = \delta \mathbb{E} \left[ \frac{dV_2(x_1 - a^*_1(x_1, \bar{F}_1), F_2)}{dF_{1,2}} \bigg|_{F'_1 = \bar{F}_1} \right], \]  

(34)

where \( dV_2(x_1 - a^*_1(x_1, \bar{F}_1), F_2)/dF_{1,2} \) is a pathwise derivative. Again by Lemma 2, it holds that
\[ \frac{dV_2(x_1 - a^*_1(x_1, \bar{F}_1), F_2)}{dF_{1,2}} = \frac{\partial V_2(x_1 - a^*_1(x_1, \bar{F}_1), F_2)}{\partial F_{2,2}} \bigg|_{F'_2 = \bar{F}_2}. \]  

(35)

For expositional convenience, we define \( x^*_2|x_1, \bar{F}_1 := x_1 - a^*_1(x_1, \bar{F}_1) \). Fixing a vector \( \bar{F}_2 \) where \( V_2((x^*_2|x_1, \bar{F}_1), F_2) \) is differentiable and substituting \( F_{2,2} \) with \( s_2 \), Lemma 3 implies
\[ \frac{\partial V_2((x^*_2|x_1, \bar{F}_1), F_2)}{\partial s_2} \bigg|_{F_2 = \bar{F}_2} = \frac{\partial v_2((x^*_2|x_1, \bar{F}_1), a^*_2((x^*_2|x_1, \bar{F}_1), F_2), F_2)}{\partial s_2} \bigg|_{F_2 = \bar{F}_2} \]
\[ = \frac{\partial p(a^*_2((x^*_2|x_1, \bar{F}_1), s_2))}{\partial s_2} \bigg|_{s_2 = \pi_2} \]
\[ + \frac{\partial W_2((x^*_2|x_1, \bar{F}_1) - a^*_2((x^*_2|x_1, \bar{F}_1), F_2), F'_2)}{\partial s_2} \bigg|_{F'_2 = \bar{F}_2} \]
\[ = (\phi^I 1\{(x^*_2|x_1, \bar{F}_1) \in \mathcal{X}^1_2(\bar{F}_2)\} + \phi^W 1\{(x^*_2|x_1, \bar{F}_1) \in \mathcal{X}^W_2(\bar{F}_2)\}) \]
\[ a^*_2((x^*_2|x_1, \bar{F}_1), \bar{F}_2). \]  

(36)

where the last equality follows from the same reasoning used to establish (27) and (28). It follows from (35) and (36) that (34) can be expressed as
\[ \frac{\partial W_1((x^*_2|x_1, \bar{F}_1), F'_1)}{\partial F_{1,2}} \bigg|_{F'_1 = \bar{F}_1} = \frac{\delta}{F_{1,2}} \mathbb{E}[(\phi^I 1\{x^*_2 \in \mathcal{X}^I_2(F_2)\} + \phi^W 1\{x^*_2 \in \mathcal{X}^W_2(F_2)\})] \]
\[ = s_2 a^*_2(x^*_2, F_2)|x_1, \bar{F}_1]. \]  

(37)

Expressions (32), (33), and (37) imply that (31) can be written as
\[ \frac{dV_1(x_1, F_1)}{dF_{0,2}} \bigg|_{F_1 = \bar{F}_1} = \frac{\delta}{F_{0,2}} \mathbb{E}[(\phi^I 1\{x^*_2 \in \mathcal{X}^I_2(F_2)\} + \phi^W 1\{x^*_2 \in \mathcal{X}^W_2(F_2)\})s_2 a^*_2(x^*_2, F_2)|x_1, \bar{F}_1]. \]  

(38)
Substituting (38) in (30) and using (31) yield

\[
\frac{\Delta_{1,2}(0, x_1, F'_0)}{\delta(0, T_1)} = \frac{\delta}{F_{0,2}} E[E[(\phi'1\{x^*_2 \in \mathcal{A}^I_2(F_2)\} + \phi^W1\{x^*_2 \in \mathcal{A}^W_2(F_2)\})]s_2a_2(x^*_2, F_2)|x_1, F_0']
\]

so that

\[
\Delta_{1,2}(0, x_1, F_0) = \frac{\delta(0, T_2)}{F_{0,2}} [E[(\phi'1\{x^*_2 \in \mathcal{A}^I_2(F_2)\} + \phi^W1\{x^*_2 \in \mathcal{A}^W_2(F_2)\})s_2a_2(x^*_2, F_2)|x_1, F_0']].
\]

The cases corresponding to \( n = 3, \ldots, N - 1 \) can be dealt with in a similar manner. \( \square \)

### B. Examples

**Example 1** (Factor hedging and equivalent hedging approaches). Suppose that \( N = 4, t = T_0 = 0, x_0 = 0 \), and that \( F_0 \) is such that doing nothing is optimal in state \((0, F_0)\) in stage 0, so that the optimal inventory level in stage 1 is 0, no matter the number of factors used.

With a three factor model, factor hedging is identical to delta hedging, that is, it consists of taking positions equal to \(-\Delta_{1,1}(0, 0, F'_0), -\Delta_{1,2}(0, 0, F'_0), -\Delta_{1,3}(0, 0, F'_0)\) in the futures contracts with maturities \( T_1, T_2, \) and \( T_3 \), respectively.

With a one factor model, consider implementing factor hedging by trading in the contract with maturity \( T_1 \). It holds that \( A_0 = \sigma_{1,1,0} \) and \( B_0 = [\sigma_{2,1,0}; \sigma_{3,1,0}] \), so that \( \gamma_{2,1,0}/|A_0| = \sigma_{2,1,0}/\sigma_{1,1,0} \) and \( \gamma_{3,1,0}/|A_0| = \sigma_{3,1,0}/\sigma_{1,1,0} \). Thus, the position in this contract is

\[
q_{1,1}(0, 0, F'_0) = \Delta_{1,1}(0, 0, F'_0) + \Delta_{1,2}(0, 0, F'_0) \frac{F'_{0,2}}{F'_{0,1}} \sigma_{2,1,0} + \Delta_{1,3}(0, 0, F'_0) \frac{F'_{0,3}}{F'_{0,1}} \sigma_{3,1,0}.
\]

Notice that the deltas in (39) are not the same as the deltas obtained with the three factor model, although our notation does not distinguish between them (the same qualification holds for the discussion to follow).

An equivalent hedging approach takes the following positions in the first two contracts:

\[
q_{1,1}(0, 0, F'_0) = \Delta_{1,1}(0, 0, F'_0) + \Delta_{1,3}(0, 0, F'_0) \frac{F'_{0,3}}{F'_{0,1}} \sigma_{3,1,0},
\]

\[
q_{1,2}(0, 0, F'_0) = \Delta_{1,2}(0, 0, F'_0).
\]

With a two factor model, suppose that factor hedging is implemented by trading in the two contracts with maturities \( T_1 \) and \( T_2 \). In this case, \( A_0 = [\sigma_{1,1,0}, \sigma_{1,2,0}; \sigma_{2,1,0}, \sigma_{2,2,0}] \) and
$B_0 = [\sigma_{3,1,0}, \sigma_{3,2,0}]$. It is easy to verify that
\[
\frac{\gamma_{3,1,0}}{|A_0|} = \frac{\sigma_{3,1,0}\sigma_{2,2,0} - \sigma_{3,2,0}\sigma_{2,1,0}}{\sigma_{1,1,0}\sigma_{2,2,0} - \sigma_{2,1,0}\sigma_{1,2,0}}, \quad (42)
\]
\[
\frac{\gamma_{3,2,0}}{|A_0|} = \frac{\sigma_{3,2,0}\sigma_{1,1,0} - \sigma_{3,1,0}\sigma_{1,2,0}}{\sigma_{1,1,0}\sigma_{2,2,0} - \sigma_{2,1,0}\sigma_{1,2,0}}. \quad (43)
\]
Hence, the relevant factor hedging positions in these contracts are
\[
q_{1,1}(0,0,F_0') = \Delta_{1,1}(0,0,F_0') + \Delta_{1,3}(0,0,F_0') F_{0,3} \frac{F_{0,3}}{F_{0,1}} \frac{\sigma_{3,1,0}}{\sigma_{1,1,0}} dF(t,T_1), \quad (44)
\]
\[
q_{1,2}(0,0,F_0') = \Delta_{1,2}(0,0,F_0') + \Delta_{1,3}(0,0,F_0') F_{0,3} \frac{F_{0,3}}{F_{0,2}} \frac{\sigma_{3,2,0}}{\sigma_{1,1,0}} dF(t,T_2), \quad (45)
\]

**Example 2 (Factor hedging with redundant factors).** As in Example 1, suppose that $N = 4$, $t = T_0 = 0$, $x_0 = 0$, and that $F_0$ is such that doing nothing is optimal in state $(0,F_0)$ in stage 0, so that the optimal inventory level in stage 1 is 0, no matter the number of factors used. While the true model has one factor, assume that hedging is done with a model that uses two factors, that is, one factor is redundant. Assuming that these models share the same loading coefficients on factor one, the redundant factor is the second one.

As in Example 1, at time 0 hedging with the two factor model is implemented by taking positions in the contracts with the two earliest maturities. These positions are given by (44)-(45). With the one factor model, hedging could be implemented by trading in the contracts with the two earliest maturities according to positions (40)-(41). Suppose that the deltas that appear in (40)-(41) and (44)-(45) are identical. Thus, the difference in the total trading cash flows corresponding to positions (40)-(41) and (44)-(45) in an infinitesimal time interval that starts at time 0 is equal to the difference between
\[
\Delta_{1,3}(0,0,F_0') F_{0,3} \frac{F_{0,3}}{F_{0,1}} \frac{\sigma_{3,1,0}}{\sigma_{1,1,0}} dF(t,T_1) \quad (46)
\]
and the sum of
\[
\Delta_{1,3}(0,0,F_0') F_{0,3} \frac{F_{0,3}}{F_{0,1}} \frac{\sigma_{3,1,0}}{\sigma_{1,1,0}} \frac{\sigma_{2,2,0} - \sigma_{3,2,0}\sigma_{2,1,0}}{\sigma_{1,1,0}\sigma_{2,2,0} - \sigma_{2,1,0}\sigma_{1,2,0}} dF(t,T_1), \quad (47)
\]
\[
\Delta_{1,3}(0,0,F_0') F_{0,3} \frac{F_{0,3}}{F_{0,2}} \frac{\sigma_{3,2,0}}{\sigma_{1,1,0}} \frac{\sigma_{1,1,0} - \sigma_{3,1,0}\sigma_{1,2,0}}{\sigma_{1,1,0}\sigma_{2,2,0} - \sigma_{2,1,0}\sigma_{1,2,0}} dF(t,T_2), \quad (48)
\]
where $dF(t,T_1)$ and $dF(t,T_2)$ are under the true one factor model. Assuming that $\Delta_{1,3}(0,0,F_0')$ is different from zero, the difference between (46) and the sum of (47) and (48) is zero whenever
\[
\frac{F_{0,3}}{F_{0,1}} \frac{\sigma_{3,1,0}}{\sigma_{1,1,0}} dF(0,T_1) = F_{0,3} \frac{F_{0,3}}{F_{0,1}} \frac{\sigma_{3,1,0}}{\sigma_{1,1,0}} \frac{\sigma_{2,2,0} - \sigma_{3,2,0}\sigma_{2,1,0}}{\sigma_{1,1,0}\sigma_{2,2,0} - \sigma_{2,1,0}\sigma_{1,2,0}} dF(0,T_1)
\]
\[+ F_{0,3} \frac{F_{0,3}}{F_{0,2}} \frac{\sigma_{3,2,0}}{\sigma_{1,1,0}} \frac{\sigma_{1,1,0} - \sigma_{3,1,0}\sigma_{1,2,0}}{\sigma_{1,1,0}\sigma_{2,2,0} - \sigma_{2,1,0}\sigma_{1,2,0}} dF(0,T_2). \quad (49)
\]
Using (42) and (43) in (17) and then expressing $dF(t, T_2)$ as

$$\frac{\sigma_{2,1,0}}{\sigma_{1,1,0}} \frac{F(t, T_2)}{F(t, T_1)} dF(t, T_1)$$

under the true model yields

$$dF(t, T_3) = \frac{F(t, T_3)}{F(t, T_1)} \left( \frac{\sigma_{3,1,0} \sigma_{2,2,0} - \sigma_{3,2,0} \sigma_{2,1,0}}{\sigma_{1,1,0} \sigma_{2,2,0} - \sigma_{2,1,0} \sigma_{1,2,0}} \right) dF(t, T_1)$$

\[= F(t, T_3) \sigma_{3,1,0} \sigma_{2,2,0} - \sigma_{2,1,0} \sigma_{1,2,0} \right) dF(t, T_1) \]

\[= F(t, T_3) + \frac{F(t, T_3)}{F(t, T_2)} \left( \frac{\sigma_{3,1,0} \sigma_{1,1,0} - \sigma_{3,1,0} \sigma_{1,2,0}}{\sigma_{1,1,0} \sigma_{2,2,0} - \sigma_{2,1,0} \sigma_{1,2,0}} \right) dF(t, T_2) \]

Letting $t$ be equal to 0 in this expression shows that (49) is true.

References

