Frequency Loyalty Programs in an Asymmetric Duopoly Market: Theoretical and Empirical Investigations

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Abstract

This paper examines frequency loyalty programs in an asymmetric duopoly market where one firm has more service locations than the other. We have analyzed the empirical data from a quasi-field experiment in a duopoly gasoline market where the larger firm initially started a full-scale loyalty program but later switched to a partial-scale loyalty program. We found that the larger firm’s loyalty program significantly increased that firm’s market share in the regular gasoline market more than in the premium gasoline market. The equilibrium analysis of a game theoretical model reveals that the optimal scale of a firm’s loyalty program shall increase with the firm’s profit margin as well as the size of the “moving” segment of consumers whose location preferences change over time (i.e., taxi drivers). The theoretical results are consistent with the empirical findings because the regular gasoline market had a sufficiently large “moving” segment while the premium gasoline market did not. Moreover, the larger firm switched from a full-scale loyalty program to a partial-scale program when both the gasoline consumption by taxi drivers as a percentage of total sales and the profit margins of the regular gasoline decreased over time.

Key words: Frequency loyalty programs, network externality, switching costs, field experiment.
1. Introduction

Consider a duopoly gasoline retail market where two competing firms owned different number of service stations. Many of the large firm’s stations were not directly contested by the small firm and enjoyed local monopoly. During a period of 4 years, the small firm always offered a frequency loyalty program. The large firm had no loyalty program for the first 17 months; offered a full-scale loyalty program (all stations participating in the program) for the next 31-month period; and then, in the 48th month, switched to a partial-scale loyalty program (only a subset of stations participating in the program). A series of regression analyses investigating the effect of the full-scale loyalty program on the large firm’s market share reveals that this program significantly increased the firm’s market share in sales of regular gasoline. However, the program had no significant effect on the firm’s market share in sales of premium gasoline. The empirical analysis, while clearly demonstrating the positive effect of the full-scale loyalty program on the large firm’s market share, raises two puzzling questions: first, why did the full-scale loyalty program have a significant effect on the firm’s market share in regular gasoline sales but not in the premium gasoline sales? Second, given the positive effect of the full-scale loyalty program, why did the firm switch to a partial-scale program?

To obtain useful insights into these two questions, we analyze a two-period duopoly game in which the larger of the two firms has multiple service stations located on a circular city market, while the small firm has only one station. The large firm can offer a full-scale program, a partial-scale program, or not offer a loyalty program. In the analysis, we consider two segments of consumers with different preference dynamics: a “moving” segment, whose preferred locations change over time, and a “constant” segment, whose preferred locations are fixed. Our analysis shows that the equilibrium scale of the large firm’s loyalty program, i.e., the number of stations included in the loyalty program, increases with the relative size of the
moving segment. This moving segment of consumers places higher value on the scale of a loyalty program because a larger network of participating stations enhances these consumers’ chances of earning points and redeeming rewards. Including those service stations not directly contested by the small firm into the loyalty program network can help the large firm acquire these moving consumers. However, the rewards offered by such uncontested stations would not increase the large firm’s sales among consumers with constant location preferences, and would merely represent costs to the firm. Firms need to balance such benefits and costs when deciding the optimal scales of their loyalty programs.

Our analysis shows that if the size of the moving segment is very large, in the equilibrium the large firm should offer a full-scale program. However, if the size of the moving segment falls in the intermediate range, the large firm should offer rewards only at stations that are in direct competition with the small firm. Finally, when the size of the moving segment is small firms may not benefit from the loyalty programs. Our analysis also shows that the equilibrium scale of the large firm’s loyalty program increases with the firm’s profit margin. We substantiate these theoretical results with the empirical evidences from the gasoline market. First, we found that the size of the moving consumer base, as measured by the number of taxi vehicles as a percentage of total vehicles, decreased over the period during which the firm switched from the full-scale program to the partial-scale program. Second, we found a decreasing trend in the profit margin of regular gasoline. Both results would favor the switch to a partial-scale loyalty program.

The scale of a loyalty program is an important decision not only to gasoline retailers. Airline companies must decide which routes to include in their frequent flier programs (FFPs) and hotel chains must decide which facilities to include in their frequent guest programs. Recently, a major Asian airline company, after acquiring a regional airline which operated only in the routes connecting mid-size cities, decided not to include the regional airline into
its frequent flier program. Alliance programs, such as Air Miles, must determine which firms
to include as program partners. Although it is feasible for an alliance program to cover all the
product categories that consumers may purchase, in practice all the alliance programs have
the limited scales and scopes. For example, utility services are seldom included in the alliance
programs. To the best of our knowledge, this is the first paper to study the optimal scale of a
loyalty program.

Our research extends the literature on the artificial compatibility principle of loyalty
programs (Cairns and Galbraith 1990, Borenstein 1996). Loyalty programs create artificial
links between otherwise independent choices. To illustrate, consider a consumer who refills
his or her gasoline tanks once a month. Each monthly gas purchase is an independent trip.
However, if gas station chains offer frequency loyalty programs, the consumer may consider
the monthly purchases to be compatible in the sense that committing to stations within the
same chain can help the consumer collect more reward points. In this situation, the loyalty
program offered by a chain with a larger number of gas stations is more attractive to the
consumer, because it provides more opportunities to earn and redeem points. Thus, positive
network externality will arise from artificial compatibility. Lederman (2003) demonstrated
this network effect in airlines’ FFPs, showing that an FFP network’s international expansion
through global alliances increased the airline’s market share in some of its domestic routes.
Our paper furthers this conclusion in two regards: First, we have identified and demonstrated
the behavior of the moving consumer segment on which the principle of artificial
compatibility is based. Second, we have shown that the largest network is not necessarily the
best. A partial-scale network can be the optimal choice for a large competitor when the
relative size of the moving segment falls into the intermediate range. Although a full-scale
program maximizes sales, the marginal sales increase that is generated from the larger scale of
the program may not justify the cost of the rewards offered by additional affiliates.
This paper adds empirical evidences to the competitive effect of loyalty programs. Most of the existing empirical studies examine the effect of loyalty programs without a proper control of the strategies of competing firms (e.g., Drèze and Hoch 1998, Lal and Bell 2003, Lewis 2004, Hartmann and Viard 2008), with a few exceptions (e.g., Lederman 2003). This paper utilizes a unique data set collected from a quasi-field experiment in an asymmetric duopoly gasoline market in which the larger of the two firms adopted a loyalty program midway through the observation period. There were no changes in the smaller firm’s loyalty program during the study period. Moreover, retail prices were regulated in this market. Thus, the data provides a more precise assessment of the effect of a frequency loyalty program.

This paper also extends our understanding of frequency loyalty programs as a price discrimination mechanism. Because heavy users are more likely to earn rewards, frequency loyalty programs may function as quantity discounts against light users (Kim, Shi, Srinivasan 2001). Hartmann and Viard (2008) present evidence of price discrimination between golfers who played at different frequencies. Even among heavy users, frequency loyalty programs may discriminate against consumers with higher costs in redeeming the rewards. Lal and Bell (2003) and Drèze and Hoch (1998) found that high-spending shoppers in frequent shopper programs often do not redeem rewards, and, more recently, Lewis (2004) found that the customers of an online grocery store responded differently to loyalty rewards. In this paper we have found that a partial-scale loyalty program can be considered a mechanism of geographic location-based price discrimination, because consumers located closer to non-participating service stations have a smaller chance of receiving rewards. We also found that premium-product users did not respond to loyalty incentives, which suggests that the frequency loyalty program influenced price discrimination between users of regular and premium gasoline.
The rest of the paper is organized as follows. First, we explain the characteristics of the market under examination, describe our data, conduct empirical analyses, and discuss the results. We then develop a theoretical model that is consistent with the industry setting, and examine the loyalty program strategies of the firms in the market. Finally, we conclude with key results and directions for future research.

2. **Empirical Analysis**

Our empirical study examines loyalty programs offered by two retail gasoline chains competing in a large Asian city. For confidentiality reasons, these two chains are denoted as firms A and B. Firm A was an incumbent in the market. Prior to firm B’s entry into the market in early 1998, firm A held a monopoly on the market and developed an extensive retail network of gasoline stations throughout the city. A chronicle of events in the market is provided in Table 1.

From the beginning of its market entry, firm B offered a full-scale frequency loyalty program. Customers gathered points for their gasoline purchases at all of firm B’s retail stations. Specifically, one point was awarded for every ten liters of fuel purchased. Firm B provided a catalogue of rewards, with a greater number of points typically required to redeem a more valuable reward. The redemptions automatically enrolled the customer in sweepstakes for greater rewards, with the winning numbers drawn at the end of each month, to encourage redemptions.

During firm B’s first 3 years in the market, firm A did not offer a loyalty program; in this time period, firm B quickly expanded its retail network and penetrated in the city’s market. By the beginning of 2001, firm B’s sales had increased to constitute a 24% share of the city’s market. In May 2002, firm A revised its strategy and began offering a full-scale frequency loyalty program similar to that of firm B; in the program, consumers could earn and
redeem points at all of firm A’s gasoline stations. In December 2004, firm A changed its reward policy and converted to a partial-scale loyalty program that excluded some of its stations. Under the new scheme, consumers could earn and redeem points only at participating stations. After the policy change, about 15% of firm A’s stations continued providing a loyalty program, making firm A’s loyalty program of a comparable size to that of firm B. Both firms A and B owned all of its stations and made marketing decisions for the entire retail chain.

Partly because of low labor costs, all gasoline stations in this area are full-service stations with workers at every pump. The retail prices were strictly regulated and set by an official petroleum pricing committee. Thus, the competing firms’ retail prices for both regular and premium gasoline were always identical.

2.1. Data and Preliminary Analysis
The data covers the period from January 2001 to December 2004 for firm A and from January 2001 to August 2005 for firm B. The data’s boundary dates are indicated by dotted lines in Table 1. In total, we collected 16 months of observations (January 2001 - April 2002) when firm A did not offer a loyalty program, and 31 months of observations (May 2002 - November 2004) when firm A offered a full-scale loyalty program. For firm B, we collected an additional 9 months of observations (December 2004 - August 2005), during which firm A offered a partial-scale program. Firm B offered a full-scale loyalty program during the entire period it was observed. For each month, the data include the number of stations in service, their monthly system-wide total sales volume, and retail prices. The data cover two lines of products: regular gasoline and premium gasoline. For firm B, we also monitored the monthly promotion expenses, including the cost of the rewards given out to consumers in exchange for
points. We also noted the total number of private vehicles from quarterly official statistics publications.

Table 2 provides descriptive statistics for firms A and B, and the characteristics of the market. During the study period, the number of stations was fairly stable for both firms. Specifically, early in the study period, the total number of firm A’s stations increased from 67 to 70, but it later dropped to 65. Firm B had a much smaller network, with a maximum of 11 stations at the beginning of the data period (2 stations were later closed). All of these variations were the result of station renovation or because of stores closing down when the neighboring area was under redevelopment. Both firms sold two grades of gasoline: regular gasoline and premium gasoline. During the observation period, there were very small variations for each firm in the number of stations selling each line of product. All of firm A’s gasoline stations sold both regular and premium gasoline. All of firm B’s stations sold regular gasoline, but some did not sell premium.

Because of its larger network of stations, firm A was clearly the market leader in terms of sales. On average, firm A sold 11,787 kiloliters of regular gas and 9,057 kiloliters of premium gas each month, and firm B sold 3,709 kiloliters of regular gas and 2,380 kiloliters of premium gas. In total sales, firm A was about three times as large as firm B. However, firm B’s sales per station were about twice as high as firm A’s (361 kiloliters of regular gas and 246 kiloliters of premium gas, and 175 kiloliters of regular and 135 kiloliters of premium, respectively). This could be due to a number of advantages enjoyed by firm B, such as larger stations, higher quality service, and an international brand image.

We measure firm A’s market share using the ratio of A’s monthly sales over the combined monthly sales of A and B, for both regular and premium products. On average, firm A had about 75% of the market share in regular gasoline and 80% of the market share in premium. We plot the time series curve for firm A’s market share for each product in Figure 1.
Interestingly, firm A’s market shares in the two gasoline grades exhibit very different patterns: while firm A’s sales initially decreased in both products, its market share of the regular gasoline increased after the implementation of the frequency loyalty program; in contrast, its share of the premium gasoline sales stayed flat.

As indicated in Table 2, the two firms’ retail prices were slightly different, with firm B’s prices, on average, about 3~5% higher than firm A’s. This may seem inconsistent with the fixed pricing policy we noted earlier; however, the price gap was created by the different accounting measurement units adopted by the two firms. ¹

City-level vehicle ownership, highway mileage, and household income all increased during the time period. From the first quarter of 2001 to the second quarter of 2005, the total number of vehicles increased from 930,940 to 1,126,000; the average household income (in local currency) increased from 6,990 to 9,650; and highway mileage increased from 304 to 517 km. The similar upward trends among these three variables indicate that they are highly correlated. Moreover, all three of these city-level variables are highly correlated with the “Month” variable and, therefore, each one of these variables can be used to capture over-time variations in market size.

2.2. Empirical Analysis and Results

Our main empirical analysis examines the effect of firm A’s loyalty program on its market share, denoted by $S_{Ajt}$ for product $j$ ($j = \text{regular or premium}$) at month $t$. The market share analysis helps separate the effect of a loyalty program from time-series trends common to both competing firms. We use a Seemingly Unrelated Regressions (SUR) model to capture

¹ Specifically, firm A measured gasoline supplies in tons, whereas firm B did so using the kiloliter. To complicate the issue, the conversion rate from ton to kiloliter changes with temperature. We used the average yearly conversion rate (which was 1.34 for regular and 1.32 for premium). The correlation between the two firms’ prices was 0.994 for regular gasoline and 0.995 for premium gasoline, a near-perfect correlation.
the potential correlations between the regular and premium gasoline sales in our study (Zellner 1962).

$$s_{jt} = \alpha_j + \beta_j \times \text{Product}_j + \gamma_j \times \text{Loyalty}_A + \epsilon_j, \quad (1)$$

where product type $j \in \{\text{regular, premium}\}$. Time variable $t$ denotes the month, ranging from 1 to 47. The independent variables $x_j$ include product specific variables (e.g., retail prices), firm-specific variables (e.g., firm B’s monthly site-promotion expenses), firm product-specific variables (e.g., the number of stations selling the product), as well as city-market variables (e.g., the number of vehicles). Firm A’s offering of a loyalty program is indicated by $\text{Loyalty}_A$, a dummy variable that becomes one for any $t \geq 17$. The loyalty program indicator is not product-specific because the program applied to all products. Finally, we assume that the vector of all error terms at month $t$, $\epsilon = (\epsilon_{\text{regular}}^t, \epsilon_{\text{premium}}^t)$ follows a joint normal distribution with zero mean, i.e.,

$$E(\epsilon_j \mid x_j) = 0, \quad (2)$$

$$E(\epsilon_j, \epsilon_s' \mid x_j) = \Omega, \quad (3)$$

where $\Omega$ is the $2 \times 2$ covariance matrix of error terms. We assume that the error terms are independent over time. As a result, $E(\epsilon_j, \epsilon_s \mid x_j, x_s) = 0$ if $t \neq s$.

The SUR model in (1) captures the effect of A’s full-scale loyalty program through a before-and-after comparison. In the absence of any explanatory variables, this model is reduced to a time-series model. Firm A’s market shares in product $j$ without a loyalty program (before) and with a full-scale loyalty program (after) are $\alpha_j^0 + \alpha_j^1 t$ and $(\alpha_j^0 + \gamma_j) + \alpha_j^1 t$, respectively. The variable $\text{Loyalty}_A$ captures the effect of firm A’s adoption of a full-scale loyalty program. Note that firm B offered a loyalty program since its market entry, so the
effect of its frequency reward program cannot be identified from the baseline market share,
\[ \alpha_j^0 + \alpha_j^1 t. \]

The candidates for the independent variables \( X_j \) include retail price and city characteristics, i.e., highway mileage, household income, and the number of vehicles. For both regular and premium gas sales, in the estimation the number of private vehicles is included as an explanatory variable. We also include firm B’s promotion expense. In our model, it is possible to correlate the sales of different grades of gasoline products through the covariance matrix, \( \Omega \). Iyer and Seetharaman (2003) consider each type of gasoline product to be a separate product market. Under this assumption, the SUR model could be analyzed as two separate equations.

We estimated the model in (1) using a feasible generalized least square method (Greene 2003), using an iterated method with a tolerance of 1e-6. Because the regression analysis involves both firm A and B, we use only the first 47 months of data in the estimations, because firm A switched to the partial-scale frequency program in the 48\textsuperscript{th} month. We present the estimation results in Table 3. Overall, the regression models performed very well with large R-squares. We also repeatedly observed a strong positive relationship between firm A’s market shares in regular and premium gasoline sales; this result makes intuitive sense, because the two grades of gas share the same brand names.

Table 3 shows that firm A’s frequency loyalty program significantly and substantially increased its market share of regular gas sales. The loyalty program increased firm A’s market share in regular gas sales, producing a net 4.2\% increase (statistically significant at \( p < 0.05 \)).

The total monthly regular gasoline sales from firms A and B during the study period were about 15,496 kiloliters, so a 4.2\% increase represents an average monthly increase of about

\footnote{We could include the interaction between the number of firm A’s stations and whether or not they offered the loyalty program to examine the network effect. Unfortunately, in our data, there were very few variations in the number of firm A’s stations, precluding a robust test.}
650 kiloliters in sales. However, firm A’s loyalty program had no significant effect on premium gas sales. These results suggest that regular gas users responded to rewards, but premium gas users did not. Finally, Table 3 shows that firm B’s promotion expenses had a significant and negative impact on firm A’s market share. Firm A’s market share in premium gasoline sales had a downward trend, as indicated by a negative coefficient for the “month” variable. Firm B may have benefited from its reputation for higher quality service.

We further confirm the positive effect of firm A’s loyalty program on regular gasoline sales by examining the relative average sales per station (A’s per station sales/B’s per station sales). As shown in Table 4, A’s regular gasoline sales per station relative to B increased by 0.07 after the implementation of a full-scale program. This increase was statistically significant. Again, the loyalty program had no significant effect on premium gas sales.

In summary, the above empirical analysis confirms that, by offering a full-scale loyalty program, firm A increased its market share significantly in the regular gasoline market. The analysis demonstrates that a large incumbent firm can benefit substantially from a full-scale loyalty program in response to an aggressive new entry. However, the analysis also creates two interesting puzzles: first, why was the incumbent firm’s loyalty program effective in the regular gasoline market but not in the premium gasoline market? Second, given the obvious success associated with the full-scale loyalty program, why did firm A switch to a partial-scale loyalty program? To seek for insights to these two puzzling questions, we next develop and analyze an asymmetric duopoly model of competition with loyalty programs.

3. Theoretical Model and Analysis

In this section, we develop and analyze a stylized theoretical game model to study the firms’ equilibrium loyalty program strategies. The model is designed with two objectives in mind: to
be complex enough to provide some useful insights into the firms’ strategies and to be consistent with the industry setting of our empirical study.

3.1. Model Formulation

In our model, we consider two service firms, denoted by A and B, competing in a circular-city market over a two-period game. The competing firms sell identical products/services. Examples of such service firms include gas stations, hotels, and coffee shops. All service stations are located along the circular city (see Figure 2), which has a circumference of one unit. To model asymmetric scales between these two firms, we assume that firm A has three stations, denoted by A₁, A₂, and A₃, whereas firm B has only one station.³ We further assume that firm A’s three stations are located equidistant from each other and firm B’s station is located between A₁ and A₂, at a distance 1/6 of a unit from both stations. The stylized structure of the model is consistent with the market in our empirical study, in which firm A, which held a monopoly before the entrance of firm B into the market, owned significantly more service stations than firm B, including stations that were not in direct competition with firm B’s stations.

The market consists of two segments of consumers. The first, denoted by c, includes α number of consumers uniformly distributed on the circle. Each consumer in this segment has a fixed location on the circle, and hence, over the two periods of the game, a constant service station preference. In contrast, the second segment, denoted by m, includes 1-α consumers with preferences that are independent between the two periods. In each period, these “moving” consumers are uniformly distributed along the circle. The design of this two-segment market (constant location vs. independent locations) is similar to that of Klemperer (1987). However, we have extended Klemperer’s structure by introducing a circular-city

³ We use the term station to refer to a retail service location. This can be broadly interpreted as a service station, store, hotel, or resort, or be generalized to refer to a service route offered by an airline or cruise, or a product within a line of products.
setting, and by allowing one firm to have multiple locations. Consumers incur a unit transport cost, \( t \), while traveling to a station on the circle. In a gasoline market, taxi drivers represent the majority of moving consumers because they can be located anywhere in the city when they need to refill their gas tanks. Finally, for both segments of consumers, we assume that, in each period, each consumer purchases one unit, and all consumers have reservation prices high enough that the market is fully covered.

Because retail gasoline prices were regulated by the government in the empirical model, we assume firms always charge an exogenously regulated retail price, \( p_k \), at period \( k \), \( k = 1, 2 \). Such price regulation is not common in other service industries; however, it allows us to focus on the firms’ decisions regarding their loyalty programs. At the beginning of period 1, the two firms simultaneously announce whether or not they will offer a loyalty program and, if a program is instituted, which service stations will participate. A loyalty program is defined as full scale when the entire service network participates and partial scale when not all service stations participate. We use the notation \{\( A_1, A_2, A_3; B \)\} to indicate the full-scale situation in which firm A has all three of its stations participating in its loyalty programs, and \{\( A_1, A_2; B \)\} to denote a partial-scale situation in which firm A only includes stations \( A_1 \) and \( A_2 \) in its loyalty program; in both cases, firm B offers a loyalty program.

A consumer can redeem a reward of amount \( r \) at the end of the second period if they have purchased gasoline from one of the firm’s participating service stations in both time periods. Note that for all of the loyalty programs, we assume a fixed reward of size \( r \), which is smaller than retail prices. This assumption is consistent with our observation that the competing firms in our study provided identical small rewards that did not change over the observation period. This assumption sufficiently simplifies the analysis to allow us to endogenously determine the scale of the loyalty programs. Singh, Jain, and Krishnan (2008) also exogenously fixed the size of reward.
Each firm makes reward decisions for all of their stations. Specifically, if firm A decides to include station $i$ in its loyalty program, then reward $r_{Ai} = r$, where station $i = 1, 2, 3$; if station $i$ is not included, $r_{Ai} = 0$. Similarly, firm B decides between $r_B = r$ and $r_B = 0$. Firm $j$ (where $j = A$ or $B$) makes reward decisions to maximize the total expected profits accrued over two time periods, given as:

$$\pi_j = \sum_{k=1}^{2} (p_j - mc_j)(S^n_j + S^c_j) - C^m_j - C^c_j$$

(4)

where $S^n_j$ and $S^c_j$ are market shares of firm $j$ in time period $k$ from segment $c$ and $m$, respectively, $C^m_j$ and $C^c_j$ are the loyalty program costs to firm $j$ from segment $c$ and $m$, respectively. Without loss of generality, we assume the marginal costs ($mc_j$) to be zero. We also assume that the reward is small enough that consumers located on one arc of the circle will not travel to another arc for service; that is, they will always choose from one of the two closest stations. Mathematically, this condition requires that $r < t/6$. In the following section, we will analyze the market share and the equilibrium results produced by the model.

### 3.2. Model Analysis: Firm’s Market Share

A firm’s market share in segment $c$ can be different from its market share in segment $m$. We began our market share analysis with segment $c$; this analysis produced to a useful property relevant to the dynamics of market shares, summarized in Lemma 1. A very similar property on the market share dynamics is identified by Kim, Shi, and Srinivasan (2001) and Singh, Jain, and Krishnan (2008).

**Lemma 1**: Each segment $c$ consumer always purchases from the same station during both of the two periods.

Proof: See Appendix 1.
Following Lemma 1, we assume that the market share of each firm is identical during each of the two periods; that is, $S_{ij}^1 = S_{ij}^2$. The market can be divided into two parts: arc $A_1BA_2$ of size $1/3$, and arc $A_1A_3A_2$ of size $2/3$. All of the consumers located on arc $A_1A_3A_2$ always buy from one of firm A’s stations (A1, A2, or A3). The consumers located on arc $A_1BA_2$ choose between two adjacent stations based on their ideal service location and the rewards stations may offer. We determine the market share of each firm in arc $A_1BA_2$ by identifying marginal consumers who are indifferent between buying from either of the firms. We sum up both firms’ sales from the entire circle and obtain their market shares in segment $c$ as follows:

$$S_{A_1}^c = S_{A_2}^c = \alpha \left( \frac{S}{6} + \frac{r_{A1} + r_{A2} - 2r_B}{4t} \right) \quad (5)$$

$$S_{B_1}^c = S_{B_2}^c = \alpha \left( \frac{1}{6} - \frac{r_{A1} + r_{A2} - 2r_B}{4t} \right) \quad (6)$$

Equations (5) and (6) imply that a firm’s market share in segment $c$ is higher when the firm offers rewards in the stations contested by the rival (A1 and A2 for firm A; B for firm B) and is lower when these rival stations offer rewards. However, the rewards offered by the non-contested station (A3 for firm A) do not affect either firm’s market share in this segment. Offering rewards at A3 merely intensifies A3’s within-network competition with A1 and A2. Therefore, increasing the network size of firm A’s loyalty program by including the non-contested station offers no competitive advantage, either in acquiring or retaining customers in segment $c$.

We now examine the market shares of the firms in segment $m$. Because these consumers’ locations vary between two periods, we have separately derived the market shares in segment $m$ in the two periods. We start with the second period and move backwards to the first period. As in segment $c$, consumers located on arc $A_1A_3A_2$ always purchase from firm A; this is true for both periods. There are a total of $2(1-\alpha)/3$ such consumers in each period. The
remaining \((1-\alpha)/3\) consumers are located on the arc \(A_1BA_2\); during the first period, \(s^m_{A1}/3\) buy from firm A and \(s^m_{B1}/3\) buy from firm B. All of these consumers choose stations based on both their own location in the second period and the reward points they may have received in the first period. Among consumers who have received reward points from one of firm A’s stations in the first period, firm A’s market shares in arc \(A_1B\) and \(BA_2\) are equal to \(\frac{1}{2} + \frac{3r_{A1}}{t}\) and \(\frac{1}{2} + \frac{3r_{A2}}{t}\), respectively. Among consumers who received reward points from firm B, firm A’s market shares in arc \(A_1B\) and \(BA_2\) are equal to \(\frac{1}{2} - \frac{3r_{B1}}{t}\) and \(\frac{1}{2} - \frac{3r_{B2}}{t}\), respectively.

Finally, among consumers who did not receive reward points from either firm in the first period, firm A’s market shares in both arc \(A_1B\) and \(BA_2\) are equal to \(\frac{1}{2}\). We denote the first-period sales in each of firm A’s stations in segment \(m\) by \(m_{As1,1}, m_{As1,2}, m_{As1,3}\). We can then express firm A’s second-period sales in segment \(m\) as follows:

\[
S^m_{A2} = \frac{2}{3}(1-\alpha) + \frac{1}{6}(r_{A1} s^m_{A1,1} + r_{A2} s^m_{A2,1} + r_{A3} s^m_{A3,1})(1 + \frac{3r_{A1} + r_{A2}}{t}) \\
+ \frac{1}{6} \left[ S^m_{A1} - (r_{A1} s^m_{A1,1} + r_{A2} s^m_{A2,1} + r_{A3} s^m_{A3,1}) \right] + \frac{1}{6} s^m_{B1} (1 - \frac{r_{B1}}{t})
\]  

(7)

where \(\frac{r_{A1}}{r} s^m_{A1,1} + \frac{r_{A2}}{r} s^m_{A2,1} + \frac{r_{A3}}{r} s^m_{A3,1}\) is the number of segment-\(m\) consumers who have received reward points from firm A in the first period. Equation (7) indicates that, given firm A’s first-period sales, firm A’s customer retention probability, and hence its second-period sales, will increase with the rewards offered by stations \(A_1\) and \(A_2\), but not the rewards offered by \(A_3\).

During the first period, all of the \(2(1-\alpha)/3\) segment-\(m\) consumers located on arc \(A_1A_3A_2\) purchase from firm A. The rest of the segment-\(m\) consumers, located on arc \(A_1BA_2\), choose between B and one of firm A’s stations. Because the competing firms always charge the same prices, consumers base their decisions on their current locations, their expected locations in the second period, and the expected rewards from the loyalty programs. To
illustrate, consider the case in which both stations A₁ and B offer rewards. We can determine the firms’ market share on arc A₁B by identifying marginal consumers who are indifferent between purchasing from A₁ and B. A key advantage of purchasing from A₁ instead of B is the higher chance to redeem rewards in the second period:

\[ s_{A_{1.2}}^m r_{A_{1}} + s_{A_{2.2}}^m r_{A_{2}} + s_{A_{3.2}}^m r_{A_{3}} = S_{B.2}^m r_{B}, \]  

(8)

where \( s_{A_{i.2}}^m \), which denotes station Aᵢ’s sales in segment m in the second period, measures the probability that a segment-m consumer who purchased from station A₁ in the first period will redeem rewards from Aᵢ in the second period (i = 1, 2, 3). Expression (8) shows that by adding A₃ to the loyalty program, the reward variable \( r_{A_{3}} \) increases from 0 to \( r \) and the chance to redeem a reward from firm A’s loyalty program in the second period increases by \( s_{A_{3.2}}^m r_{A_{3}} \).

Thus, adding A₃ to firm A’s loyalty program network can help the firm acquire the customers in segment m who are located on arc A₁B during the first period. Under the probability \( s_{A_{3.2}}^m \) these consumers will purchase from station A₃ in the second period. In this case, their reward points earned from station A₁ or A₂ in the first period can be redeemed if and only if A₃ is a participating station. These consumers will consider the reward points earned at A₁ or A₂ to be more valuable, and hence are more likely to purchase from firm A. This is how the artificial network effect works.

3.3. Model Analysis: Equilibrium Results

In response to firm B’s entry to the market, firm A must decide whether or not to offer a frequency loyalty program, and which of its stations should participate. We address these problems in a formal analysis of pure strategy equilibrium. Given the two-period nature of our model, we derive the subgame perfect equilibrium using the backward induction method. For each given parameter space we obtain a unique pure strategy equilibrium. Before presenting
our results in a formal proposition, we first state some equilibrium properties in the following two lemmas.

**Lemma 2:** It is suboptimal that station A3 provides rewards while stations A1 and A2 do not.

**Lemma 3 (Symmetry between A1 and A2)** Loyalty outcomes \{A1; B\}, \{A2; B\}, \{A1, A3; B\}, and \{A2, A3; B\} cannot be equilibrium outcomes.

Proof: See Appendix 2.

Lemma 2 states that, when choosing participating stations for its loyalty program, firm A should first consider stations A1 and A2. Because these two stations are in direct competition with firm B’s station, the rewards offered by A1 or A2 are more effective in competing for customers than the same rewards offered by A3. Station A3 is located in an area not contested by firm B and, as such, all of the consumers located on arc A1A3A2 are locked into firm A. Adding station A3 to the loyalty program, while useful to segment-m consumers, merely increases firm A’s reward costs in serving the segment-c consumers, without their producing any overall increase in sales. Lemma 3 states a symmetry property between stations A1 and A2. When firm B offers a loyalty program, if firm A gains profit from offering rewards at either station A1 or A2, then firm A is better off offering rewards at both A1 and A2. This is simply because A1 and A2 are symmetrically located in the market. Using the equilibrium analysis, we can obtain firms’ profit functions under different loyalty program scales. Comparing the equilibrium profits under different conditions, we reach the following proposition:

**Proposition 1**

1.1. There exists parameter spaces where either \{A1, A2, A3; B\} or \{A1, A2; B\} is the pure-strategy equilibrium.

1.2. When firm B provides rewards, if \((1 - \alpha) > \frac{6t^2(r + t - p_1 - p_2)}{3t^2 - 11p^2 + (12p_1 + 16p_2 + 29r)p - 36r^2 + 30r^2p_2} \), firm A is better off providing a partial-scale loyalty program \{A1, A2\} than no rewards.
1.3. When firm B provides rewards, if
\[
(1 - \alpha) > \frac{6(2_1)}{1 + (2_1) (2_2 + 2_3 - 3_2)}
\]
and \(-4r^2 + (12 p_1 + 12 p_2 - 17 r)^2 + (63 r - 18 p_1 + 6 p_2) r + 108 r^3 - 72 r^2 p_3 > 0\), firm A is better off providing a full-scale loyalty program \{A_1, A_2, A_3\} than a partial-scale reward program \{A_1, A_2\}.

1.4. When firm B provides rewards, the lower the profit margin, the more likely firm A will prefer a partial-scale loyalty program \{A_1, A_2\} over a full-scale loyalty program \{A_1, A_2, A_3\}.

Proof: Please see Appendix 3.

The proposition contains four results. Proposition 1.1 establishes that both offering a partial-scale and a full-scale loyalty program can be equilibrium outcomes. We provide the specific conditions in Appendix 3. Overall, the smaller firm (firm B) has a greater incentive to offer a loyalty program because it has fewer locked-in customers. When firms offer loyalty programs to attract marginal consumers, these locked-in customers become free riders for the rewards, resulting in a loss of profits. In our model, the larger firm (firm A) has more such profits to lose and thus has less incentive to provide a loyalty program.

Propositions 1.2 and 1.3 characterize firm A’s optimal loyalty program strategy in response to firm B’s loyalty program. More specifically, Proposition 1.2 identifies the conditions under which firm A prefers a partial-scale program \{A_1, A_2\} over not offering any program. Proposition 1.3 identifies the conditions under which firm A prefers a full-scale program \{A_1, A_2, A_3\} over a partial-scale one \{A_1, A_2\}. Combined, these two propositions delineate the parameter spaces in which programs of different scales are optimal. The key conditions can be expressed in terms of the relative size of the moving segment (\(\alpha\)). Specifically, with everything else being equal, a larger segment \(m\) makes it more likely that firm A will offer a loyalty program of any kind; when segment \(m\) is very large, firm A will offer a full-scale program rather than a partial-scale program. Station A_3 joins the loyalty program if and only if segment \(m\) is large enough because the network size of A’s loyalty
program has no impact on segment c consumers. Station A₃ provides a positive network effect that enables a consumer with changing location preferences to earn and redeem points in more locations. However, because station A₃ is not contested by firm B’s station, any rewards offered to segment c consumers located near A₃ represent marketing costs without return. To decide whether station A₃ should participate in the loyalty program, firm A has to compare the benefit of a positive network effect in acquiring segment m consumers with the cost of the rewards handed out to segment c consumers. Clearly, the size of segment m is a critical factor that can tip the balance. Finally, Proposition 1.4 states that when the retail margins are low, firm A’s benefit from gaining extra market share in segment m is small. As a result, the partial-scale loyalty program is more likely to be preferred.

We illustrate the main results of Proposition 1 in Figure 3, where the horizontal axis indicates the sum of profit margins (p₁+p₂) and the vertical axis represents the size of segment c (α). For other parameters, we set r = 0.1 and t = 0.65. (We have experimented with a large set of values for r and t and have found very similar patterns.) We numerically solve firm A’s optimal loyalty program decisions for each set of parameter values. The results in Figure 4 are very consistent with Proposition 1. First, if the values of all other parameters are unchanged, the optimal scale of firm A’s loyalty program is smaller when the size of the constant-location segment (m) is larger (and hence the size of moving segment (α) is smaller). This is consistent with Propositions 1.2 and 1.3 combined. Second, given that the values of all other parameters are unchanged, the optimal scale of firm A’s loyalty program increases with the total profit margins from the two periods. More specifically, when the total profit margin is high, firm A will benefit most from offering a full-scale program over a partial-scale program. This is consistent with Proposition 1.4.
3.4. Empirical Evidences

We now return to the gasoline market to seek for empirical evidences substantiating the theoretical results in Proposition 1. A critical component of the theoretical model is the presence of a “moving” segment. We identify the taxi drivers as a representative group of such consumers. Taxi drivers travel around the city to meet and drop the customers. It is a common knowledge that taxi drivers typically purchase regular gasoline. During the data period, the city government did not issue any new taxi plates and the number of taxi licenses issued did not change, remaining constant at approximately 30,000. The vast majority of taxi drivers did not own their own taxis (i.e., the plates and vehicles), but paid a monthly lease fee to a taxi management company and paid for their own fuel.

*Did taxi drivers respond to firm A’s loyalty program?*

To answer this question, we obtained data from a taxi audit study conducted by the industry. The study audited the total number of vehicles visiting firm A’s stations in six selected weeks; two weeks from before the implementation of the full-scale loyalty program, and four weeks from after. Figure 4 shows the weekly number of taxis entering firm A’s stations during these weeks. We can see an immediate and dramatic increase in the number of taxis visiting the stations after May 18, 2002, when the loyalty program was implemented at all of firm A’s stations. The average number of daily taxi visits increased from 12,182 during the week of January 26, 2002, to 15,251 during the week of May 18, 2002, an increase of more than 20%. From this 6-week study, we estimated that the average number of daily visits increased by 2,436. Unfortunately, we do not know if there were changes in the average gasoline purchase per trip. An expert estimated that an average of 10 liters of gasoline were purchased during each taxi visit. If this amount remained constant over the 6-week study period, the increase in daily gasoline sales would be approximately 24 kiloliters, representing an increase of about
720 kiloliters per month. This estimate is consistent with our regression analysis. The results confirm that taxis as the “moving” segment accounted for the main economic effect of firm A’s loyalty program.

*Did firm A’s full-scale loyalty program increase its profit?*

Loyalty programs are costly promotional tools for companies, and these costs could often deter a company from providing loyalty incentives to their customers (Dowling and Uncles 1997, Lal and Bell 2003). It is therefore important to investigate the impact of reward programs on firms’ profitability in addition to sales. During the first 6 months after firm A implemented its full-scale program, the company distributed 10,115,000 reward points worth $1,011,000 (all currency refers to local currency). Among these reward points, 5,649,000 were redeemed, for a redemption rate of about 55.8%. The total cost of the loyalty program to firm A, including the cost of rewards, the cost of membership cards, advertising fees, and legal fees, was $659,000. As estimated in Table 3, firm A’s regular gasoline sales increased about 650 kiloliters each month, which produces to a 6-month sales increase of about 4,906 kiloliters. From the data, we estimated an average margin of about $0.44 per liter. Thus, the loyalty program led to an incremental profit of $4,906,000 × $0.44 = $1.72 million. Clearly, without accounting for initial set-up costs, firm A’s loyalty program had a very positive impact on profit.

*Why did firm A changed from a full-scale to a partial-scale loyalty program?*

Our theoretical proposition provides two possible answers to the puzzling switch of firm A from a full-scale to a partial-scale loyalty program. First, as the number of moving consumers as a percentage of entire market decreases, firm A’s incentive to provide rewards in non-contested stations would also decrease. As a result, firm A would resort to a partial-scale
loyalty program. In the market under study, the number of taxi drivers in our market remained constant during the study period, but the total number of vehicles increased steadily from 930,940 to 1,126,000. Although the number of taxi vehicles made up a small percentage of the total vehicles, taxis accounted for a very substantial amount of gasoline sales.\(^4\) Thus, as indicated by our Propositions 1.2 and 1.3, firm A should offer a reward program. However, the percentage of gasoline sales accounted by the taxi vehicles decreased over time as private vehicle ownership increased. As implied by our Propositions 1.2 and 1.3, the decrease in the relative size of the moving segment indicates a reduced incentive for firm A to offer rewards in those stations not contested by firm B (station A3 in the model). This may explain why firm A switched to a partial-scale loyalty program.

Second, Proposition 1.4 indicates that a shift from a full-scale to a partial-scale loyalty program can also be caused by a reduced profit margin. In Figure 5, we plot firm A’s price/cost ratio over 48 months and see a clear downward trend after May 2002, when firm A implemented its full-scale program. Our data on firm A ends in December 2004, when firm A changed to a partial-scale loyalty program. We must therefore use firm B’s profit margin to conduct a longer trend analysis covering the period after December 2004. The simple regression analysis in Table 5 shows that the profit margin for regular gasoline significantly decreased over time, but the time trend for premium gasoline was not statistically significant. As the profit margin for regular gas decreased, it was no longer profitable to attract additional moving customers by offering rewards at non-contested service stations like A3. Thus, our theoretical prediction from Proposition 1.4 is also consistent with empirical observation.

4. Conclusion

\(^4\) In 2002 the taxi drivers accounted for 39.7% of the total visits in firm B’s stations.
This paper studies the effectiveness of frequency loyalty programs in an asymmetric duopoly retail gasoline market. We find that a firm’s frequency loyalty program had a significant and positive effect on its market share of regular gasoline sales but not premium gas sales. Using empirical evidence and a tailored theoretical model consistent to the market we study, we demonstrate that this result was due to the presence of a large segment of taxi drivers who typically purchased regular gasoline. These taxi drivers were constantly traveling and hence their location preferences were independent over time. These consumers value a loyalty program in a large network to facilitate earning and redeeming reward points. In this market, the number of moving consumers as a percentage of the entire market and profit margins both decreased over time, prompting the large competitor to switch from a full-scale to a partial-scale program. A partial-scale loyalty program allowed the firm to price discriminate based on service locations and extract more surplus from the consumers located close to stations not contested by the rival firm.

This paper is the first study to theoretically and empirically investigate the optimal scale of a frequency loyalty program, an issue that is applicable to any firms and alliances that sell multiple products/services or serve customers at different locations. We have theoretically examined the equilibrium scale of loyalty programs, and have empirically studied the effect of a key determining factor, namely, the relative size of a consumer segment with changing location preferences. In our empirical study, we could not directly test the network effect because of insufficient variation in the number of gas stations. Future empirical research will be needed to address this issue. In our theoretical model, we assume price and reward are exogenous based on our observation of the gasoline retail market we study. This allows us to focus on the key issue in the paper: the optimal scale of loyalty programs. However, future research might examine the impact of these two restricting assumptions on optimal scale of loyalty programs although we do not believe it will change our results substantively.
References


Figure 1: Firm A’s market share, during the period from January 2001 to November 2004, by product
Market share = A’s Sales/(A’s sales + B’s Sales)

Figure 2: A two-segment circular-city market
Figure 3: Optimal program scale for firm A
Figure 4: Total weekly taxi visits at firm A

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<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>12172</td>
<td>85204</td>
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<td>26/1–2/2, 2002</td>
<td>11752</td>
<td>11577</td>
<td>11450</td>
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Beginning of full-scale loyalty program
Figure 5: Price/cost ratio for firm A
Table 1: Chronicle of Events

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<tr>
<th>Date</th>
<th>Firm A:</th>
<th>Firm B:</th>
</tr>
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<tr>
<td>Prior to January 2001</td>
<td>Held a monopoly until 1998; no loyalty program offered</td>
<td>Entered the market in 1998; offered a full-scale loyalty program</td>
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<td>January 2001</td>
<td>Beginning of our data set</td>
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<td>May 2002</td>
<td>Firm A: Started a full-scale loyalty program</td>
<td>Firm B: Full-scale loyalty program</td>
</tr>
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<td>December 2004</td>
<td>Firm A: Changed to a partial-scale loyalty program</td>
<td>Firm B: Full-scale loyalty program</td>
</tr>
<tr>
<td></td>
<td>End of data</td>
<td></td>
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<tr>
<td>August 2005</td>
<td>Firm A: Partial-scale loyalty program</td>
<td>Firm B: Full-scale loyalty program</td>
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<tr>
<td></td>
<td>No data</td>
<td>End of our data</td>
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Table 2: Descriptive Statistics

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<th>Mean</th>
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<td></td>
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<tr>
<td>Number of stations</td>
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<td>65</td>
<td>70</td>
<td>67.44</td>
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<td>Sales volume (kilocliter)</td>
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<td></td>
</tr>
<tr>
<td>Regular</td>
<td>48</td>
<td>8,622</td>
<td>14,234</td>
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<td>13,700</td>
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<td><strong>Firm B</strong></td>
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<tr>
<td>Number of stations</td>
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<td>477.1</td>
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<td>86.582</td>
<td>65.210</td>
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<tr>
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<td>1,126</td>
<td>1,011</td>
<td>60.446</td>
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<td>32,000</td>
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Table 3: Seemingly Unrelated Regressions (SUR) analysis of firm A’s market share during the period from January 2001 to November 2004

Firm A’s market share = A’s Sales/(A’s sales + B’s Sales)

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<thead>
<tr>
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<td></td>
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<td>t-value</td>
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<td>Month</td>
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<td>R-square</td>
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<table>
<thead>
<tr>
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<th>A R/B R</th>
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<td>System R-square</td>
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<td>Correlation matrix of residue</td>
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<tr>
<td>A R/B R</td>
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<td>AR/BR</td>
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<tr>
<td>A P/B P</td>
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<td>AP/Bp</td>
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<td>0.8286</td>
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**: Significant at 0.01 level.
*: Significant at 0.05 level.
Table 4: Seemingly Unrelated Regressions (SUR) analysis of relative average sales per station during the period from January 2001 to November 2004
Relative average sales = A’s sales/B’s sales

<table>
<thead>
<tr>
<th></th>
<th>Per-station regular gas sales ratio A/B</th>
<th>Per-station premium gas sales ratio A/B</th>
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<tr>
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<td>Estimate</td>
<td>t-value</td>
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<tr>
<td>Intercept</td>
<td>0.809*</td>
<td>2.219</td>
</tr>
<tr>
<td>Number of vehicles</td>
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</tr>
<tr>
<td>Loyalty A</td>
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<tr>
<td>B’s site promotion cost</td>
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<td>-2.715</td>
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</table>

Correlation matrix of residue

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<th>A R/B R</th>
<th>A P/B P</th>
</tr>
</thead>
<tbody>
<tr>
<td>A R/B R</td>
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<td>0.694</td>
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<td>A P/B P</td>
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</table>

**: Significant at 0.01 level.
*: Significant at 0.05 level.

Table 5: Time Trend of Firm B’s Profit Margin

<table>
<thead>
<tr>
<th></th>
<th>Firm B’s Profit Margin on Regular Gasoline</th>
<th>Firm B’s Profit Margin on Premium Gasoline</th>
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</thead>
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<tr>
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<td>Coefficient</td>
<td>t-stats</td>
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<td>Constant</td>
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<tr>
<td>Month</td>
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<td>-2.21</td>
</tr>
</tbody>
</table>

R-square 8.30%  0.04%
Technical Appendices

Appendix 1: Proof for Lemma 1

We analyze consumers at different locations separately.

(1) Consumers on arc A1–B
In the first period, a consumer (with reservation price \( \theta \)) located at a distance of \( X \) from A1 on arc A1–B compares the total surpluses from buying from A1 twice (2\((-p_1-p_2-2Xt/6+r_{A1})\)) and total surpluses from buying from B twice (2\((-p_1-p_2-2(1-X)t/6+r_{B})\)). If this consumer switches from one firm to another in two periods, the consumer’s surplus will be 2\((-p_1-p_2-t/6, \) which is clearly worse than buying from the same firm twice. Thus, the consumer always prefers to buy twice from the same firm, either A or B, rather than from two different firms in two periods. The consumer buys from A1 in the first period if

\[
2Xt/6+r_{A1} \geq 2(1-X)t/6+r_{B}.
\]

In this case, in the second period, this consumer will again purchase from A1 because

\[
\theta - p_2 - \frac{t}{6}X + r_{A1} > \theta - p_2 - \frac{t}{6}(1-X).
\]

However, if

\[
\theta - p_2 - \frac{t}{6}X + \frac{r_{A1}}{2} < -\frac{t}{6}(1-X) + \frac{r_{B}}{2},
\]

the consumer buys from B in the first period and again from B in the second period, because

\[
\theta - p_2 - \frac{t}{6}X < \theta - p_2 - \frac{t}{6}(1-X) + r_{B}.
\]

(2) Consumers on arc B–A2
The proof for consumers on arc B–A2 is the same as above (1).

(3) Consumers on arc A1–A3
A consumer prefers to buy from A1 over A3 in the first period if

\[
\frac{t}{3}X + \frac{r_{A1}}{2} \geq -\frac{t}{3}(1-X) + \frac{r_{A3}}{2}.
\]

In the second period, the consumer compares the surplus from A1, which is

\[
\theta - p_2 - \frac{t}{3}X + r_{A1},
\]

to the surplus from A3, which is

\[
\theta - p_2 - \frac{t}{3}(1-X) + r_{A3}.
\]

If both A1 and A3 provide rewards, that is, \( r_{A1} = r_{A3} \), then the consumer will buy from A1 because

\[
\theta - p_2 - \frac{t}{3}X \geq \theta - p_2 - \frac{t}{3}(1-X).
\]

If only one of the two stations provides rewards, the consumer will also buy from A1 in the second period. Similarly, the consumer prefers to buy from A3 over A1 in the first period if
\[-\frac{t}{3}X + \frac{r_{A1}}{2} < -\frac{t}{3}(1 - X) + \frac{r_{A3}}{2}, \text{ and will buy from A3 again in the second period because}\]

\[\theta - p_{2} - \frac{t}{3}X + r_{A1} < \theta - p_{2} - \frac{t}{3}(1 - X) + r_{A3}.\]

(4) Consumers on arc A3–A2
The proof for the consumers on arc A3–A2 is the same as in (3).

This completes the proof for Lemma 1.

**Appendix 2: Proof for Lemma 3**
We separately prove that each of the loyalty outcomes \{A_{1}; B\}, \{A_{2}; B\}, \{A_{1}, A_{3}; B\}, and \{A_{2}, A_{3}; B\} cannot be an equilibrium outcome.

1. \{A_{1} (or A_{2}); B\} is a dominated strategy.
   (1). (B) If only firm B offers rewards, firm A’s sales will be
   \[
   \pi_{A}^{m} = ap_{1}\left(\frac{5}{6} - \frac{r}{2t}\right) + ap_{2}\left(\frac{5}{6} - \frac{r}{2t}\right),
   \]  
   (A2a.1)

   \[
   \pi_{A}^{n} = (1 - \alpha)p_{1}\left(\frac{5}{6} - \frac{r}{2t} - \frac{r^{2}}{2t^{2}}\right) + (1 - \alpha)p_{2}\left(\frac{5}{6} - \frac{r}{6t} - \frac{r}{6t^{2}} - \frac{r^{3}}{2t^{3}}\right).
   \]  
   (A2a.2)

(2). (A1 or A2, B) If firm B and A1 (or A2) provide rewards, firm A’s sales will be:
   \[
   \pi_{A}^{m} = ap_{1}\left(\frac{5}{6} - \frac{r}{4t}\right) + ap_{2}\left(\frac{5}{6} - \frac{r}{4t}\right) - ar\left(\frac{1}{4} + \frac{r}{4t}\right),
   \]  
   (A2a.3)

   \[
   \pi_{A}^{n} = (1 - \alpha)p_{1}\left(\frac{5}{6} - \frac{r}{24t} - \frac{r^{3}}{24t^{3}}\right) + (1 - \alpha)p_{2}\left(\frac{5}{6} - \frac{r}{24t} + \frac{r}{24t^{2}} - \frac{r^{3}}{8t^{3}}\right) - (1 - \alpha)r\left(\frac{1}{16} + \frac{7r}{24t} + \frac{11r^{2}}{48t^{2}} + \frac{r^{3}}{4t^{3}}\right).
   \]  
   (A2a.4)

(3). (A1, A2; B) If B, A1 and A2 provide reward, firm A’s sales will be:
   \[
   \pi_{A}^{m} = ap_{1}\left(\frac{5}{6}\right) + ap_{2}\left(\frac{5}{6}\right) - ar\left(\frac{1}{4} + \frac{r}{2t}\right),
   \]  
   (A2a.5)

   \[
   \pi_{A}^{n} = (1 - \alpha)p_{1}\left(\frac{5}{6} + \frac{r}{3t} + \frac{r^{2}}{2t^{2}}\right) + (1 - \alpha)p_{2}\left(\frac{5}{6} + \frac{7r^{2}}{6t^{2}} + \frac{r^{3}}{t^{3}}\right) - (1 - \alpha)r\left(\frac{1}{16} + \frac{7r}{12t} + \frac{29r^{2}}{12t^{2}} + \frac{3r^{3}}{t^{3}}\right).
   \]  
   (A2a.6)

Next, we check the profit difference.
Firm A’s profit difference between \{A_{1} (or A_{2}); B\} and \{B\}:
\[ \Delta \pi_4 = ap \left( \frac{r}{4t} \right) + ap \left( \frac{r}{4t} \right) - ar \left( \frac{1}{4} + \frac{r}{4t} \right), \]  
(A2a.7)

\[ \Delta \pi_3 = (1 - a)p_1 \left( \frac{r}{8t} \right) + (1 - a)p_2 \left( \frac{r}{8t} \right) + \frac{5r^2}{24t} + \frac{3r^3}{8r} - (1 - a)r \left( \frac{1}{16} + \frac{7r}{24t} + \frac{11r^2}{48t} + \frac{r^3}{4t} \right). \]  
(A2a.8)

Firm A’s profit difference between \{A1, A2; B\} and \{A1; B\}

\[ \Delta \pi_4 = ap \left( \frac{r}{4t} \right) + ap \left( \frac{r}{4t} \right) - ar \left( \frac{1}{4} + \frac{r}{4t} \right), \]  
(A2a.9)

\[ \Delta \pi_3 = (1 - a)p_1 \left( \frac{3r}{8t} \right) + (1 - a)p_2 \left( \frac{3r}{8t} \right) + \frac{9r^2}{8r} + \frac{17r^3}{8r} - (1 - a)r \left( \frac{3}{16} + \frac{7r}{24t} + \frac{35r^2}{16t} + \frac{11r^3}{4t} \right). \]  
(A2a.10)

We want to show that if firm A can make more profits in \{A1 (or A2); B\} than in \{B\}, firm A can then make more profits in \{A1, A2; B\} than in \{A1 (or A2), B\}. The net gains (increase in profits) are the same in these two scenarios because equations A2a.7 and A2a.9 are identical. Thus, we only need to compare the net gains from segment m. We find that equation A2a.8 is bigger than 3 times equation A2a.10. In this case, we only need to show that equation A2a.10 is positive.

Note that if firm A makes more profits in \{A1 (or A2); B\} than in \{B\}, at least one of equations A2a.7 and A2a.8 will be positive. If we assume equation A2a.7 is positive, then

\[ \frac{r}{t} > \frac{p_1 + p_2 - r}{t}. \]

Substituting this into equation A2a.8, we get:

\[ (1 - a)p_1 \left( \frac{r}{8t} \right) + (1 - a)p_2 \left( \frac{r}{8t} \right) + \frac{5r^2}{24t} + \frac{3r^3}{8r} - (1 - a)r \left( \frac{1}{16} + \frac{7r}{24t} + \frac{11r^2}{48t} + \frac{r^3}{4t} \right). \]

\[ > (1 - a) \left( \frac{r}{p_1 + p_2 - r} \left( \frac{p_1}{8} + \frac{r}{4t} \right) + \frac{5r}{24t} + \frac{3r^2}{8r} - (p_1 + p_2 - r) \left( \frac{1}{16} + \frac{7r}{24t} + \frac{11r^2}{48t} + \frac{r^3}{4t} \right) \right) > 0 \]  
(A2a.11)

If firm A makes more profits in \{A1 (or A2); B\} than in \{B\}, equation A2a.8 is positive. This finishes this proof that \{A1 (or A2); B\} is a dominated strategy.

(2). \{A1 (or A2), A3; B\} is a dominated strategy.

Firm A always makes more profits in \{A1, A2; B\} than in \{A1 (or A2), A3;B\}

\[ \Delta \pi_3 = \frac{r}{t} p_1 \left( \frac{5 + a}{24t} + \frac{(1 - a)3r}{4t} \right) + p_1 \left( \frac{5 + a}{24t} + \frac{(1 - a)9r}{24t} + \frac{(1 - a)17r^3}{8t} \right) \]

\[ - ar \left( \frac{1}{12} + \frac{r}{4t} - \frac{(1 - a)r}{13} \right) > 0 \]  
(A2a.12)

Thus, \{A1 (or A2), A3; B\} is a dominated strategy.
Appendix 3: Proof for Proposition 1

Proof for Proposition 1.1:

(1) \{A1, A2, A3; B\} is an equilibrium outcome if the following conditions hold:

(i) Firm A makes more profits in \{A1, A2, A3; B\} than in \{A1, A2; B\}

\[
\begin{align*}
\Delta \pi_A^1 &= \frac{r}{t} \left( p_1 (1 - \alpha ) \left( \frac{r}{2t} \right) + p_2 (1 - \alpha ) \left( \frac{r}{6t} - \frac{2r^2}{t^2} \right) \right) \\
&\quad - ar \left( \frac{1}{3} \cdot \frac{r}{2t} \right) - (1 - \alpha ) r \left( \frac{4}{9} \cdot \frac{r}{36t} - \frac{7r^2}{4t^2} - \frac{3r^3}{t^3} \right) > 0
\end{align*}
\]

\( (A3a.1) \)

(ii) Firm A makes more profits in \{A1, A2, A3; B\} than in \{B\}

\[
\begin{align*}
\Delta \pi_A^2 &= \frac{r}{t} \left( p_1 \left( \frac{5 - 2\alpha}{6} + \frac{(1 - \alpha)r}{2t} \right) + p_2 \left( \frac{5 - 2\alpha}{6} + \frac{(1 - \alpha)3r}{2t} + \frac{(1 - \alpha)r^2}{2t} \right) \right) \\
&\quad - ar \left( \frac{5}{6} - (1 - \alpha )r \left( \frac{25}{36} + \frac{25r}{18t} + \frac{2r^2}{3t^2} \right) \right) > 0
\end{align*}
\]

\( (A3a.2) \)

(iii) Firm B makes more profits in \{A1, A2, A3; B\} than \{A1, A2; A3\}

\[
\begin{align*}
\Delta \pi_B^3 &= \frac{r}{t} \left( p_1 \left( \frac{1 + 2\alpha}{6} + \frac{(1 - \alpha)r}{2t} \right) + p_2 \left( \frac{1 + 2\alpha}{6} - \frac{(1 - \alpha)r}{2t} + \frac{(1 - \alpha)r^2}{2t^2} \right) \right) \\
&\quad - ar \left( \frac{1}{6} \cdot \frac{r}{36t} - \frac{2r^2}{3t^2} \right) > 0
\end{align*}
\]

\( (A3a.3) \)

If equation A3a.2 holds, we get

\[
\frac{r}{t} > \frac{ar \left( \frac{5}{6} + \frac{(1 - \alpha)r}{2t} \right) \left( \frac{25}{36} + \frac{25r}{18t} + \frac{2r^2}{3t^2} \right)}{p_1 \left( \frac{5 - 2\alpha}{6} + \frac{(1 - \alpha)r}{2t} \right) + p_2 \left( \frac{5 - 2\alpha}{6} + \frac{(1 - \alpha)3r}{2t} + \frac{(1 - \alpha)r^2}{2t^2} \right)}
\]

\( (A3a.2b) \)

Substitute this into equation A3a.3, we get:

\[
\begin{align*}
\Delta \pi_B^3 &= \frac{r}{t} \left( \frac{5 - 2\alpha}{6} + \frac{(1 - \alpha)r}{2t} \right) + p_2 \left( \frac{5 - 2\alpha}{6} + \frac{(1 - \alpha)3r}{2t} + \frac{(1 - \alpha)r^2}{2t^2} \right) \\
&\quad - ar \left( \frac{1}{6} \cdot \frac{r}{36t} - \frac{2r^2}{3t^2} \right) > 0
\end{align*}
\]

\( (A3a.3b) \)

It can be shown that the right hand side is always positive. Thus, if (ii) holds, (iii) is true as well.

(iv) Firm A makes more profits in \{A1, A2, A3; B\} than \{A1, A3; B\}

\[
\begin{align*}
\Delta \pi_A^4 &= \frac{r}{t} \left( p_1 \left( \frac{13 - 7\alpha}{24} + \frac{(1 - \alpha)r}{4t} \right) + p_2 \left( \frac{13 - 7\alpha}{24} + \frac{(1 - \alpha)23r}{24t} + \frac{(1 - \alpha)r^2}{8t^2} \right) \right) \\
&\quad - ar \left( \frac{1}{4} \cdot \frac{r}{4t} \right) - (1 - \alpha )r \left( \frac{51}{144} + \frac{37r^2}{72t} + \frac{r^3}{48t^2} - \frac{r^3}{4t^2} \right) > 0
\end{align*}
\]

\( (A3a.4) \)

Similarly, if equation A3a.2 holds, we get
\[
\frac{r}{t} > \frac{\alpha r}{p_t} \left( \frac{5}{6} - \frac{2\alpha}{5} + \frac{(1 - \alpha)(r(25r + 25r^2)}{36} + \frac{25r}{18r} + \frac{2r^2}{3r^2} \right)
\]

Put this into equation A3a.4, we get:

\[
\Delta \pi^*_A > \frac{\alpha r}{p_t} \left( \frac{5}{6} - \frac{2\alpha}{5} + \frac{(1 - \alpha)(r(25r + 25r^2)}{36} + \frac{25r}{18r} + \frac{2r^2}{3r^2} \right)
\]

It can be shown that the right hand side is always positive. Thus, if (ii) holds, (iv) is also true.

(v) Firm A makes more profits in \{A1, A2, A3; B\} than in \{A1 (or A2); B\}

\[
\Delta \pi^*_A = \frac{r}{t} p_t(17 - 11\alpha) + \frac{(1 - \alpha)r}{4t} + \frac{17 - 11\alpha}{24t} + \frac{(1 - \alpha)(r^2)}{24t} - \frac{(1 - \alpha)r}{8t^2}
\]

Then we get:

\[
\frac{r}{t} > \frac{\alpha r}{p_t} \left( \frac{5}{6} - \frac{2\alpha}{5} + \frac{(1 - \alpha)(r(25r + 25r^2)}{36} + \frac{25r}{18r} + \frac{2r^2}{3r^2} \right)
\]

Substitute this into equation A3a.2, we get:

\[
\Delta \pi^*_A > \frac{\alpha r}{p_t} \left( \frac{5}{6} - \frac{2\alpha}{5} + \frac{(1 - \alpha)(r(25r + 25r^2)}{36} + \frac{25r}{18r} + \frac{2r^2}{3r^2} \right)
\]

It can be shown the right hand side is always positive. Thus if (v) holds, (ii) is also true.

If equations A3.1 and A3.5 hold, the equilibrium exits.

\[
\zeta_t p_t(1 - \alpha) + \frac{r}{t} \zeta_p(1 - \alpha) \frac{1}{3} + \frac{r}{t} \zeta_p(1 - \alpha) \frac{1}{3} + \frac{2r^2}{3r^2} \geq \alpha r \left( \frac{1}{3} - \frac{1}{2} \right) + \frac{1}{2} - \frac{1}{2} r \frac{7}{9} - \frac{7}{9} r \frac{3r}{4r} - \frac{3r}{4r}
\]

\[
\zeta_t p_t(1 - \alpha) \frac{17}{24} + \zeta_p(1 - \alpha) \frac{17}{24} + \zeta_p(1 - \alpha) \frac{17}{24} + \zeta_p(1 - \alpha) \frac{17}{24} \geq \alpha r \left( \frac{1}{3} - \frac{1}{2} \right) + \frac{1}{2} - \frac{1}{2} r \frac{7}{9} - \frac{7}{9} r \frac{3r}{4r} - \frac{3r}{4r}
\]

(2) \{A1, A2; B\} is the equilibrium if:

(i) Firm A makes more profits in \{A1, A2; B\} than in \{B\}
\[
\Delta \pi_A^* = \frac{r}{t} p_i \left( \frac{1 - \alpha}{} + \frac{(1 - \alpha) r}{2t} \right) + \frac{r}{t} p_j \left( \frac{1 - \alpha}{} + \frac{(1 - \alpha) 4r}{3t} + \frac{(1 - \alpha) 5r^2}{2t^2} \right)
- \alpha r \left( \frac{1}{2} \right) \frac{r}{2t} (1 - \alpha) r \left( \frac{1 + 17 r}{4} + \frac{29 r^2}{12t} + \frac{3 r^3}{12t^2} \right) > 0
\]  
(A3b.1)

(ii) Firm A makes more profits in \{A1, A2; B\} than in \{A1, A2, A3; B\}

\[
\Delta \pi_A^* = \frac{r}{t} p_i \left( \frac{1 - \alpha}{} + \frac{1 - \alpha}{} \left( \frac{1}{3} \right) \frac{r}{2t} \right) + \frac{r}{t} p_j \left( \frac{1 - \alpha}{} + \frac{(1 - \alpha) r}{6t} \right)
- \alpha r \left( \frac{1}{3} \right) \frac{r}{6t} (1 - \alpha) r \left( \frac{1}{9} + \frac{4 r}{36t} - \frac{7 r^2}{4t^2} - \frac{3 r^3}{t^3} \right) < 0
\]  
(A3b.2)

(iii) Firm B makes more profits in \{A1, A2; B\} than in \{A1, A2\}

\[
\Delta \pi_B = \frac{r}{t} p_i \left( \frac{1 + 2 \alpha}{} + \frac{(1 - \alpha) r}{6t} \right) + \frac{r}{t} p_j \left( \frac{1 + 2 \alpha}{} - \frac{(1 - \alpha) r}{6t} \right)
- \alpha r \left( \frac{1}{6} \right) \frac{r}{6t} (1 - \alpha) r \left( \frac{1}{12} + \frac{5 r^2}{3t} - \frac{3 r^3}{2t^2} \right) > 0
\]  
(A3b.3)

(iv) Firm A makes more profits in \{A1, A2; B\} than \{A1; B\}

\[
\Delta \pi_A^* = \frac{r}{t} p_i \left( \frac{3 - \alpha}{} + \frac{(1 - \alpha) r}{8} \right) + \frac{r}{t} p_j \left( \frac{3 - \alpha}{} - \frac{(1 - \alpha) r}{8} \right)
- \alpha r \left( \frac{1}{4} \right) \frac{r}{4t} (1 - \alpha) r \left( \frac{3}{16} + \frac{7 r^2}{4t} + \frac{11 r^3}{4t^2} \right) > 0
\]  
(A3b.4)

(v) Firm A makes more profit in \{A1, A2; B\} than \{A1, A3; B\}

\[
\Delta \pi_A^* = \frac{r}{t} p_i \left( \frac{5 + \alpha}{} + \frac{(1 - \alpha) 3r}{24} \right) + \frac{r}{t} p_j \left( \frac{5 + \alpha}{} - \frac{(1 - \alpha) 3r}{24} \right)
- \alpha r \left( \frac{1}{12} \right) \frac{r}{12t} (1 - \alpha) r \left( \frac{13}{144} + \frac{25 r^2}{24t} + \frac{35 r^3}{4t^2} \right) > 0
\]  
(A3b.5)

Equation A3b.5 always holds. Thus, when equations A3b.1, A3b.2, A3b.3, and A3b.4 hold, \{A1, A2; B\} is an equilibrium.

\[
\frac{r}{t} p_i \left( \frac{1}{2} + \frac{(1 - \alpha) r}{2t} \right) + \frac{r}{t} p_j \left( \frac{1}{2} - \frac{(1 - \alpha) 4r}{2t} + \frac{(1 - \alpha) 5r^2}{2t^2} \right) > \alpha r \left( \frac{1}{2} \right) \frac{r}{2t} (1 - \alpha) r \left( \frac{1}{4} + \frac{17 r}{12t} + \frac{29 r^2}{12t^2} + \frac{3 r^3}{t^3} \right)
\]  
(A3b.e1)

\[
\frac{r}{t} p_i \left( \frac{1 + 2 \alpha}{} + \frac{(1 - \alpha) r}{6t} \right) + \frac{r}{t} p_j \left( \frac{1 + 2 \alpha}{} - \frac{(1 - \alpha) 4r}{6t} + \frac{(1 - \alpha) 5r^2}{6t^2} \right) < \alpha r \left( \frac{1}{3} \right) \frac{r}{6t} (1 - \alpha) r \left( \frac{4}{9} + \frac{17 r}{36t} + \frac{29 r^2}{36t^2} + \frac{3 r^3}{4t^2} \right)
\]  
(A3b.e2)

\[
\frac{r}{t} p_i \left( \frac{3 - \alpha}{} + \frac{(1 - \alpha) 3r}{8t} \right) + \frac{r}{t} p_j \left( \frac{3 - \alpha}{} - \frac{(1 - \alpha) 3r}{8t} \right) > \alpha r \left( \frac{1}{4} \right) \frac{r}{4t} (1 - \alpha) r \left( \frac{3}{16} + \frac{7 r^2}{24t} + \frac{35 r^3}{4t^2} + \frac{11 r^4}{4t^3} \right)
\]  
(A3b.e3)

\[
\frac{r}{t} p_i \left( \frac{5 + \alpha}{} + \frac{(1 - \alpha) 3r}{24t} \right) + \frac{r}{t} p_j \left( \frac{5 + \alpha}{} - \frac{(1 - \alpha) 3r}{24t} \right) > \alpha r \left( \frac{1}{12} \right) \frac{r}{12t} (1 - \alpha) r \left( \frac{13}{144} + \frac{25 r^2}{24t} + \frac{35 r^3}{4t^2} \right)
\]  
(A3b.e4)

**Proof for Proposition 1.2:**

When firm B provides rewards, firm A prefers a partial-scale reward program over no rewards if condition (A3b.1) holds. We can simplify condition (A3b.1) to:

\[
[3r^3 - 11rt^2 + (12 p_i + 16 p_j - 29 r)rt - 36 r^3 + 30 r^2 p_j](1 - \alpha )
\]

\[
> 6t^2 (r + t - p_i - p_j)
\]  
(A3c.1)
Because \( t > 6r \) and \( \vartheta_i = [3t^3 - 11t^2 + (12p_1 + 16p_2 - 29r)rt - 36t^3 + 30r^2p_3] > 0 \), condition (A3c.1) becomes:

\[
(1 - \alpha) > \frac{6r^2(r + t - p_1 - p_2)}{3t^3 - 11t^2 + (12p_1 + 16p_2 - 29r)rt - 36t^3 + 30r^2p_3}
\]  

(A3c.2)

**Proof for Proposition 1.3 and 1.4:**

When firm B provides rewards, firm A prefers a full-scale reward program over a partial scale program if condition (A3a.1) holds. We simplify condition (A3a.1) to:

\[
\frac{-4t^3 + (12p_1 + 12p_2 - 17r)t^2 + (63r - 18p_1 + 6p_2)rt + 108r^3 - 72r^2p_3}{6t^2(2t - 3r)} (1 - \alpha)
\]  

(A3c.3)

Set \( \vartheta_{2a} = [-4t^3 + (12p_1 + 12p_2 - 17r)t^2 + (63r - 18p_1 + 6p_2)rt + 108r^3 - 72r^2p_3] \)

and \( \vartheta_{2b} = 6t^2(2t - 3r) \).

Because \( r < (t/6) \), we have \( \vartheta_{2b} > 0 \). If \( \vartheta_{2a} < 0 \), then \( \vartheta_{2a}(1 - \alpha) < \vartheta_{2b} \), and a partial-scale program is always preferred. If \( \vartheta_{2a} > 0 \), when \( (1 - \alpha) > \frac{\vartheta_{2a}}{\vartheta_{2a}} = \vartheta_{2ab} \), the full-scale reward program is preferred.

It can be easily shown that when retail prices (profit margins) are bigger, \( \vartheta_{2a} \) is more likely to be bigger than zero.