A maritime inventory routing problem:

Discrete time formulations and valid inequalities

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Abstract

A single product maritime inventory routing problem in which the production and consumption rates vary over the planning horizon is studied. The problem includes a heterogeneous fleet and multiple production and consumption ports with limited storage capacity.

Two discrete time formulations are developed. One is a standard formulation and the other is a fixed charge network flow reformulation. Mixed integer sets arising from the decomposition of the formulations are identified. Several lot-sizing relaxations are derived for the network formulation and used to establish valid inequalities to strengthen the proposed formulations.

Considering a set of instances based on real data, a computational study is conducted to test the formulations and the effectiveness of the inclusion of valid inequalities. By using a branch and bound scheme based on the strengthened network formulation a large majority of the instances with up to sixty time periods are solved to optimality.

Keywords: Inventory routing, maritime transportation, mixed integer linear formulation, lot-sizing relaxations.

1 Introduction

Maritime transportation is a major mode of transportation covering more than 80% of the world trade by volume, UNCTAD \(^[30]\). Large quantities are transported over long distances, and often inventories exist at the loading or discharge ports of the sailing legs. When one actor or cooperating actors in the maritime supply chain have the responsibility of both the transportation of goods and the inventories at the ports, the underlying planning problem is a maritime inventory routing problem (MIRP). Such problems are very complex, but a modest improvement in the fleet utilization and loading/discharge quantities can translate into large profit improvements due to a

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capital intensive industry. This means that there is a great potential and need for research in the area of MIRPs.

The problem analyzed in this paper is a single product MIRP. The product is produced at loading (production) ports and consumed at discharge (consumption) ports. It is possible to store the product in inventories with time dependent capacities at both types of ports. The production and consumption rates are deterministic but may vary over the planning horizon. There are berth capacities at the ports, limiting the number of ships that can load or discharge at the same time.

A heterogeneous fleet of ships is used to transport the product. Each ship has a given capacity, speed, and loading/discharge rate. The ships can wait outside a port before entering for a loading or discharge operation. A ship can both load and discharge at multiple ports in succession. The initial position and load on board each ship is known at the beginning of the planning horizon.

The sailing costs, waiting costs and port costs are all ship dependent. The planning problem is to design routes and schedules for the fleet that minimize the transportation and port costs and determine the load or discharge quantity at each port visit without exceeding the storage capacities. Depending on the segment the fleet is operating in, the typical planning period spans from one week up to several months.

Maritime inventory routing problems have achieved increasing attention in the literature the last decade; see the surveys on MIRPs in Andersson et al. [3] and Christiansen and Fagerholt [5] and general reviews on ship routing and scheduling by Christiansen et al. [7] and Christiansen et al. [8]. Most of the published contributions are based on real cases from the industry due to a demand for support when taking complex routing and inventory management decisions. Similar to the problem analyzed here, many of the studies describe single product MIRPs; see for instance Christiansen [4] and Flatberg et al. [11] considering ammonia supply chains, Furman et al. [12] focusing on the transportation of oil products and Grønhaug et al. [15] discussing liquefied natural gas (LNG) distribution. However, several cases are described in the literature where multiple products need to be taken into account; see for instance Ronen [23], Al-Khayyal and Hwang [1], Rakke et al. [22], Siswanto et al. [27] and Christiansen et al. [6]. For several of the problems described, the production and/or consumption rates are considered given and fixed during the planning horizon; see for instance Christiansen [4], Al-Khayyal and Hwang [1], and Siswanto et al. [27]. For the underlying models, time continuous variables are often used. In Ronen [23], Grønhaug and Christiansen [14], Grønhaug et al. [15], Rakke et al. [22], Furman et al. [12], and Song and Furman [28] time discrete models are developed to overcome the complicating factors with variable production and consumption rates as are also considered in this paper. As discussed in both Andersson et al. [3] and Song and Furman [28], most combined maritime routing and inventory management problems described in the literature are a particular version of the MIRP and tailor-
made methods are developed to solve the problem. These methods are often based on heuristics or decomposition techniques. The choice of these solution approaches might be explained by the high complexity of real MIRPs and the opportunity to utilize the special structure of the problem. However, constant hardware developments combined with the theoretical advances in optimization techniques have produced optimization solvers capable of handling increasingly larger instances. Currently, it is possible to obtain optimal or near optimal solutions to small real instances occurring in maritime transportation problems using commercial solvers, see Agra et al. [2], [25], and [26].

The study of valid inequalities for mixed integer sets and the derivation of extended formulations is currently receiving large attention with several applications to other mixed integer problems such as in routing and lot-sizing problems. However, little work has been done on applying these techniques to maritime transportation problems. A few contributions already exist within the field and the attention to this area is increasing. Sherali et al. [24] included valid inequalities in order to strengthen the formulations of an oil products transportation problem, and Persson and Göthe-Lundgren [18] developed valid inequalities within a column generation approach for a combined MIRP and production scheduling problem. Also, Grønhaug et al. [15] include valid inequalities to improve the path flow formulation presented for an LNG inventory routing problem. Finally, Song and Furman [28] present valid inequalities for MIRPs including several practical constraints for solving problems in different shipping segments. A time discrete model tightened with valid inequalities is given for an Inventory Routing problem in [29].

The objective of this research has been to study a general MIRP with time varying production and consumption rates and to develop tight mixed integer linear programming formulations for the problem. Therefore the paper starts with an intuitive formulation, called the standard formulation, for the MIRP studied and then a stronger formulation which is a fixed charge network flow formulation (FCNF) is presented. In addition, valid inequalities for the problem are developed that are based on known families of valid inequalities from the lot-sizing literature. Several of these valid inequalities can potentially be used for other inventory routing problems and in tailor-made solution approaches such as column generation to solve even larger instances than those presented here.

The remainder of the paper is organized as follows. Section 2 presents the two alternative mixed integer linear formulations for the MIRP. In Sections 3 and 4 several mixed integer relaxations are derived for the network formulation. These relaxations are used to develop valid inequalities to strengthen the proposed formulation. Section 5 presents the computational study. Some concluding remarks follow in Section 6.
2 Problem formulations

To formulate the problem as a mixed integer linear program, a number of modeling decisions have been made. The first consideration is whether to work with continuous or discrete time periods. Since both production and consumption rates may vary over the planning horizon, a discrete time formulation is proposed. It is therefore assumed that the waiting time, the time for loading and discharge and the sailing times can be measured as an integer multiple of a basic time period, such as 8 hours or a day.

In each time period, a ship can be either waiting, in operation (loading or discharge), or sailing. Two assumptions are made: i) a ship does not visit a port without carrying out an operation, and ii) waiting always takes place on arrival at a port before any operations start. The first is natural while the second can in certain not very likely circumstances limit the possible solutions. We discuss at the end of Section 2 how the models can be adapted if these assumptions are dropped.

The assumptions imply that if a ship operates (loads or discharges) in a port in one time period, it can either continue to operate at that port or sail to another port in the next time period. It cannot wait in a port and immediately after, sail to another port. This also means that if a ship waits outside a port in one time period, it can either continue waiting or start operating in the port in the next time period, but it cannot sail to another port before it has operated.

The movement of a ship is illustrated in the time expanded network in Figure 1. The ship starts at its initial position O and sails to Port 1. At Port 1 the ship operates for two periods (period 4 and 5) before sailing to Port 3 where it waits for one period before operating. The ship then sails to Port 2 where it waits and operates before it ends its schedule. For modeling purposes it is assumed that the ship then sails to an artificial end node D. The sailing to this node is marked with a dashed line in Figure 1. Each path through the network defines a schedule for the ship. A schedule consists of a geographical route, i.e. a sequence of ports, and the time periods when the ship operates at the ports.

In Section 2.1 a mixed integer linear formulation of the problem is given. This formulation has some similarities to other formulations of MIRPs; see for instance Song and Furman [28] and Grønhaug and Christiansen [14], and will henceforth be called the standard formulation. The standard formulation is then reformulated as a fixed charge network flow problem in Section 2.2 - the main difference lies in the precision with which the load on each ship is modeled which leads to a tighter linear programming relaxation.

2.1 Standard formulation

To model the problem as a mixed integer linear program, the following notation is introduced
Figure 1: Example of the movement of a ship in a time expanded network. The arc labels are $O$ for operating, $W$ for waiting and $S$ for sailing.

Sets
- $N^P$ set of production ports with indices $i$ and $j$,
- $N^D$ set of consumption ports with indices $i$ and $j$,
- $N$ set of production and consumption ports with indices $i$ and $j$, $N = N^P \cup N^D$,
- $T$ set of time periods with index $t$,
- $V$ set of ships with index $v$.

Parameters
- $B_{it}$ berth capacity in number of ships at port $i$ in time period $t$,
- $C_{ijv}$ sailing cost from port $i$ to port $j$ with ship $v$,
- $C_v^W$ waiting cost for ship $v$ per time period,
- $C_{iv}^P$ port cost at port $i$ for ship $v$ per time period,
- $D_{it}$ consumption at port $i$ in period $t$,
- $P_{it}$ production at port $i$ in period $t$,
- $K_v$ capacity of ship $v$,
- $L_{iv}^0$ initial load on board ship $v$,
- $Q_v$ the upper bound on the amount ship $v$ loads/discharges per time period,
- $\overline{S}_i^t$ the upper bound on the inventory level at port $i$ at the end of time period $t$,
- $\underline{S}_i^t$ the lower bound on the inventory level at port $i$ at the end of time period $t$,
- $S_{iy}^0$ the inventory level in port $i$ at the beginning of the planning horizon,
- $o(v)$ the initial position for ship $v$,
- $d(v)$ the artificial end node for ship $v$,
- $T_{ijv}$ the sailing time from port $i$ to port $j$ for ship $v$. 
Variables

\( o_{ivt} \) 1 if ship \( v \) operates in port \( i \) in time period \( t \), 0 otherwise,

\( x_{ijvt} \) 1 if ship \( v \) leaves port \( i \) in period \( t \) en route for port \( j \) (This means that the ship was operating in port \( i \) in time period \( t \)), 0 otherwise,

\( w_{ivt} \) 1 if ship \( v \) is waiting outside port \( i \) in time period \( t \), 0 otherwise,

\( l_{vt} \) load on board ship \( v \) at the end of time period \( t \),

\( q_{ivt} \) the quantity loaded/discharged in time period \( t \) at port \( i \) by ship \( v \),

\( s_{it} \) inventory level in port \( i \) at the end of time period \( t \).

The problem can now be modeled as follows

\[
\min \sum_{v \in V} \sum_{i \in N \cup \{o(v)\}} \sum_{j \in N \cup \{d(v)\}} \sum_{t \in T} C_{ijv}^T x_{ijvt} + \sum_{v \in V} \sum_{i \in N} \sum_{t \in T} C_{iv}^P o_{ivt} + \sum_{v \in V} \sum_{i \in N} \sum_{t \in T} C_{iv}^W w_{ivt}, \quad (1)
\]
subject to:

\[
\sum_{j \in N \cup \{d(v)\}} \sum_{t \in T} x_{o(v) jv t} = 1, \quad \forall v \in V, \quad (2)
\]

\[
\sum_{i \in N \cup \{o(v)\}} \sum_{t \in T} x_{id(v) it} = 1, \quad \forall v \in V, \quad (3)
\]

\[
\sum_{j \in N \cup \{o(v)\}} x_{jiv, t - T_jv} + w_{iv, t - 1} + o_{iv, t - 1} =
\]

\[
\sum_{j \in N \cup \{d(v)\}} x_{ijv t} + w_{ivt} + o_{ivt}, \quad \forall v \in V, i \in N, t \in T, \quad (4)
\]

\[
o_{iv, t - 1} \leq \sum_{j \in N \cup \{d(v)\}} x_{ijv t} + o_{ivt}, \quad \forall v \in V, i \in N, t \in T, \quad (5)
\]

\[
o_{iv, t - 1} \geq \sum_{j \in N \cup \{d(v)\}} x_{ijv t}, \quad \forall v \in V, i \in N, t \in T, \quad (6)
\]

\[
\sum_{v \in V} o_{ivt} \leq B_{it}, \quad \forall i \in N, t \in T, \quad (7)
\]

\[
0 \leq q_{ivt} \leq Q_v o_{ivt}, \quad \forall v \in V, i \in N, t \in T, \quad (8)
\]

\[
s_{i, t - 1} + \sum_{v \in V} q_{ivt} = D_{it} + s_{it}, \quad \forall i \in N^D : t \in T, \quad (9)
\]

\[
s_{i, t - 1} + P_{it} = \sum_{v \in V} q_{ivt} + s_{it}, \quad \forall i \in N^P : t \in T, \quad (10)
\]

\[
S_{it} \leq s_{it} \leq \underline{S}_{it}, \quad \forall i \in N, t \in T, \quad (11)
\]

\[
s_{i0} = S^0_{i}, \quad \forall i \in N, \quad (12)
\]

\[
l_{v, t - 1} + \sum_{i \in N^P} q_{ivt} - \sum_{i \in N^D} q_{ivt} - l_{vt} = 0, \quad \forall v \in V, t \in T, \quad (13)
\]

\[
0 \leq l_{vt} \leq K_v, \quad \forall v \in V, t \in T, \quad (14)
\]

\[
l_{v0} = L^0_v, \quad \forall v \in V, \quad (15)
\]

\[
x_{ijv t} \in \{0, 1\}, \quad \forall v \in V, i \in N \cup \{o(v)\}, \quad (16)
\]

\[
o_{ivt}, w_{ivt} \in \{0, 1\}, \quad \forall v \in V, i \in N, t \in T. \quad (17)
\]

The objective function (1) is the sum of all sailing costs, operating costs and waiting costs.

Constraints (2) and (3) ensure that each ship starts and finishes a schedule. Note that a ship can be idle the whole planning horizon by sailing directly from the initial node to the artificial end node. Constraints (4) are the ship flow conservation constraints at each port in each period. Constraints (5) prevent a ship from waiting at a port after an operation, while constraints (6) make sure that a ship can only sail after operating. The berth capacities are stated in constraints (7). Constraints (8) ensure that a ship cannot load/discharge if it is not in operating mode and defines the upper bound on the quantity loaded/discharged. The inventory balances for consumption and
production ports are expressed in constraints (9) and (10) respectively. Constraints (11) and (12) define the upper and lower inventory limits and the initial inventory. Constraints (13), (14), and (15) guarantee the equilibrium of the quantity on board the ship. All binary variable restrictions are stated in constraints (16) and (17).

The connections between the variables in the formulation are shown in Figure 2. Ship \( v \) arrives at port \( i \) at the start of time period 2 and waits for one period before discharge. The figure shows the movement of the ship through the \( x, w, \) and \( o \) variables, the quantity discharged from the ship, the \( q \) variables, the external demand, the \( D \) parameters, and the inventory levels at a discharge port, the \( s \) variables. Note that \( q_{ivt} \) can only be positive if the ship is in loading/discharge mode, i.e. \( o_{ivt} = 1 \), \( q_{iv1} \) and \( q_{iv2} \) are therefore zero and not marked with bold arrows.

\[
\begin{align*}
\text{Figure 2: Discharge operation at port } i \text{ for ship } v. 
\end{align*}
\]

### 2.2 Fixed charge network flow formulation

As the linear programming bounds provided by formulation (1) – (17) are weak, it is natural to try to strengthen the formulation. One way to do this is provided by the observation that the problem can be viewed as a single commodity fixed charge network flow problem in which the commodity is supplied externally at loading ports, flows along the arcs corresponding to the ships routes before being deposited at the discharge ports where it can satisfy the external demands.

This cannot be modeled in the network similar to that presented in Figure 2 since there it is possible for a ship to wait throughout its visit to a port and not operate at all. Thus we have chosen to model the problem as a single commodity fixed charge network flow problem. This allow us to take advantage of known inequalities for such problems. To keep the FCNF structure, an extended network is needed in which each arc representing either waiting or operating is split into one arc representing waiting and another arc representing operating. In addition each node in the upper layer in Figure 2 is split into one node in which a ship can enter the port and one node in which it can depart from the port.

Since a ship can only depart from a port after an operation, it is also necessary to distinguish
between the first time the ship operates during each call to a port and the following operating periods. Thus new nodes and arcs are introduced along with the corresponding binary arc and flow variables in order to model the operations of the ship: $o^A_{ivt}$ indicates whether ship $v$ starts to operate at port $i$ in period $t$ and $o^B_{ivt}$ indicates the succeeding operations at that port.

Figure 3 illustrates the extended network corresponding to the situations shown in Figure 2. The ship has arrived at port $i$ at the beginning of period 2, waits in period 2, starts operating (unloading) in period 3, continues operating in period 4 and then leaves for port $k$ in period 5. The ship can only depart from the second layer, so it is forced to operate at least once.

With the new $o^A_{ivt}, o^B_{ivt}$ variables, the ship flow conservation constraints (4) now can be formulated as

$$
\sum_{j \in N \cup \{o(v)\}} x_{jiv,t-T_{jiv}} + w_{iv,t-1} = w_{ivt} + o^A_{ivt} \quad \forall v \in V, i \in N, t \in T, \quad (18)
$$

$$
o^A_{iv,t-1} + o^B_{iv,t-1} = o^B_{ivt} + \sum_{j \in N \cup \{d(v)\}} x_{ijvt} \quad \forall v \in V, i \in N, t \in T, \quad (19)
$$

$$
o^A_{ivt}, o^B_{ivt} \in \{0, 1\}, \quad \forall v \in V, i \in N, t \in T, \quad (20)
$$

and together with constraints (2), (3), (16), and (17) they describe the movement of the ships through the extended network given in Figure 3.

The coordination between the path of the ships and the loading or discharge of the product in a port is provided by the constraints

$$
o^A_{ivt} + o^B_{ivt} = o_{ivt} \quad \forall v \in V, i \in N, t \in T, \quad (21)
$$
which also provide the link between the old and the new operating variables.

To complete the fixed charge network flow formulation, the variable \( l_{vt} \) and the constraints (13) – (15) describing the quantity on board the ships are replaced by flow variables and flow conservation constraints. A flow variable is defined for each arc in the extended network.

- \( f^X_{ijvt} \): load on board ship \( v \) when traveling from port \( i \) to port \( j \), leaving at time period \( t \),
- \( f^{OA}_{vt} \): load on board ship \( v \) before operating at port \( i \) in time period \( t \)
  for the first time during the current visit,
- \( f^{OB}_{vt} \): load on board ship \( v \) before continuing to operate at port \( i \) in time period \( t \)
  after an operation in time period \( t - 1 \),
- \( f^W_{vt} \): load on board ship \( v \) while waiting during time period \( t \) at port \( i \).

Hence, \( f^X_{ijvt} \), \( f^{OA}_{vt} \), \( f^{OB}_{vt} \) and \( f^W_{vt} \) represent the flow on the arcs defined by the binary variables \( x_{ijvt} \), \( a^A_{vt} \), \( a^B_{vt} \) and \( w_{vt} \), respectively. This leads to the flow conservation constraints:

\[
\sum_{j \in N \setminus \{o(v)\}} f^X_{ijvt} - T_{jiv} + f^W_{iv,t-1} - f^{OA}_{vt} - f^{OB}_{vt} = 0 \quad \forall v \in V, i, t \in T, (22)
\]

\[
f^{OA}_{vt} + f^{OB}_{vt} + f^W_{vt} = \sum_{j \in N \setminus \{d(v)\}} f^X_{ijvt} \quad \forall v \in V, i \in N^P \cup \{o(v)\}, t \in T, (23)
\]

\[
f^{OA}_{vt} + f^{OB}_{vt} - q^w_{vt} = f^W_{vt} + \sum_{j \in N \setminus \{d(v)\}} f^X_{ijvt} \quad \forall v \in V, i \in N^D \cup \{o(v)\}, t \in T, (24)
\]

\[
f^X_{o(v),jt} = f^0_{v,o(v)jt} \quad \forall v \in V, j \in N \cup \{d(v)\}, t \in T, (25)
\]

and the variable upper bound and nonnegativity constraints:

\[
0 \leq f^X_{ijvt} \leq K_v x_{ijvt} \quad \forall v \in V, i \in N \cup \{o(v)\}, j \in N \cup \{d(v)\}, t \in T, (26)
\]

\[
0 \leq f^{OA}_{vt} \leq K_v a^A_{vt} \quad \forall v \in V, i \in N, t \in T, (27)
\]

\[
0 \leq f^{OB}_{vt} \leq K_v a^B_{vt} \quad \forall v \in V, i \in N, t \in T, (28)
\]

\[
0 \leq q^w_{vt} \leq Q_v a^w_{vt} \quad \forall v \in V, i \in N, t \in T, (29)
\]

\[
0 \leq f^W_{vt} \leq K_v w_{vt} \quad \forall v \in V, i \in N, t \in T. (30)
\]

The FCNF formulation is defined by (1) – (3), (7) – (12), (16), (17), and (18) – (30). We denote by \( \mathbb{X}^{\text{FCNF}} \) the set of feasible solutions of the FCNF model.

The standard formulation can be related to the FCNF formulation as follows: (4) – (6) are replaced by (18) – (21) and (13) – (15) are replaced by (22) – (30). It can also be shown that constraints (4) – (6) and (13) – (15) are valid for FCNF, so that FCNF is a stronger formulation than the standard formulation.

**Example 2.1** Consider the following instance for \( T = \{1, \ldots, 30\} \), with one loading port \( N^P = \{1\} \), one discharge port \( N^D = \{2\} \), and a single ship (we thus omit the index \( v \) from variables and parameters). Assume the initial position \( o(v) \) coincides with Port 1. Let \( B_{it} = 1, \forall t \in T \),
The optimal solution has cost 162. An optimal route is given by $x_{o(v),1,1} = x_{1,2,6} = x_{1,2,18} = x_{2,1,12} = x_{2,d(v),24} = 1$. There are two loading operations, in periods 5 and 17, and two unloading operations, in period 11 and 23, all of them at the maximum load/unload level of 50.

Using the standard formulation, the value of the linear relaxation is 12. This cost results mainly from the operations, since the transportation cost is very low because the routing variables are $x_{o(v),1,6} = 0.08, x_{o(v),2,1} = 0.08$. From period 6 to period 30, the ship simultaneously loads 4 units at port 1 and unloads 4 units at port 2. All the travel variables between the two ports are zero. This happens because constraints (13) only ensure the equilibrium onboard the ship. There are no flow constraints linking each unit loaded at Port 1 to the same unit unloaded at Port 2.

The linear relaxation of the FCNF model has value 160. In this case, there are many fractional routing variables (that for brevity we omit their values here) that ensure the connection between the two ports since the flow constraint forces any unit unloaded at Port 2 to have been loaded at Port 1.

In Figure 4 the graph corresponding to loading port $i$ and ship $v$ is depicted. The two top layers model the ship operations while the third layer is for the port inventory. If there is more than one ship, then the two top layers must be replicated with one such network for each ship. The aggregate arriving and departing flows $f_{A}^{X}_{ivt}$ and $f_{B}^{X}_{ivt}$ are introduced to ease the presentation.

Remark 2.2 If the initial model assumptions are dropped, i.e. obliging a ship to operate at least once during a visit to a port and imposing that a ship only waits before arrival at a port, it suffices to replace the equality (21) by the inequality $o_{ivt} \leq o_{ivt}^{A} + o_{ivt}^{B}$. Now periods in which $o_{ivt}^{A} + o_{ivt}^{B} = 1$ and $o_{ivt} = 0$ are waiting periods, so the cost term $C_{W}^{v}(o_{ivt}^{A} + o_{ivt}^{B} - o_{ivt})$ must be added to the objective function. If a ship is forced to operate at least once, the constraint $q_{ivt} \geq Q o_{ivt}$ is added where $Q > 0$ is an appropriate minimum load/unload amount can be added.

3 Strengthening the fixed charge network flow formulation

The FCNF formulation can be tightened by adding inequalities that are valid inequalities for mixed integer sets derived as relaxations of the FCNF formulation. In this section several such relaxations are identified while Section 5 shows how the addition of valid inequalities for these relaxations can be very important in solving the test instances. The relaxations can be grouped into two major types: mixed integer relaxations resulting from single row relaxations along with simple or variable
bound constraints, such as knapsack sets or single row mixed integer sets, and lot-sizing relaxations which can be regarded as sets with more structure. The sets of valid inequalities for different relaxations may overlap. In Pochet and Wolsey [21] a comprehensive study of valid inequalities and reformulations for the mixed integer sets used in this paper is given. Some inequalities that are discussed in this section, such as knapsack inequalities, are also valid or can be easily adapted for the standard formulation.

3.1 Mixed integer relaxations

For each port, simple mixed integer relaxations are obtained from bounding the flow across cut-sets separating the given port from the remaining ports in the FCNF network.

Loading ports

The idea here is to look at the flow in and out of loading port \( i \) over a given time interval. Define the time interval \( \mathbf{T} = [k, \ell] \subseteq T \). For each ship \( v \), define a set \( \mathbf{T}_v \subseteq \mathbf{T} \) representing a subset of the time periods in \( \mathbf{T} \) in which ship \( v \) is assumed to operate at port \( i \). Also define \( \mathbf{T}_v^+ = \{ t \in \mathbf{T}_v : t + 1 \notin \mathbf{T}_v \} \) as the time periods in \( \mathbf{T} \) followed immediately by the departure of ship \( v \) from port \( i \) and \( \mathbf{T}_v^- = \{ t \in \mathbf{T}_v : t - 1 \notin \mathbf{T}_v \} \) as the time periods in \( \mathbf{T} \) in which ship \( v \) starts operating at \( i \).

Summing the flow conservation constraints (24) for load port \( i \) over all ships \( v \in V \) and time
periods \( t - 1 \in T_v \), gives

\[
\sum_{v \in V} \sum_{t \in T_v} q_{ivt} = \sum_{v \in V} \sum_{t \in T_v} (f_{iv,t+1}^{OB} - f_{ivt}^{OB}) + \sum_{v \in V} \sum_{j \in N \cup \{d(v)\}} \sum_{t \in T_v} f_{ijv,t+1}^X - \sum_{v \in V} \sum_{t \in T_v} f_{ivt}^{OA}.
\]

Using

\[
\sum_{v \in V} \sum_{t \in T_v} (f_{iv,t+1}^{OB} - f_{ivt}^{OB}) = \sum_{v \in V} \sum_{t \in T_v} f_{iv,t+1}^{OB} - \sum_{v \in V} \sum_{t \in T_v} f_{ivt}^{OB}
\]

and nonnegativity of \( f_{ivt}^{OA} \) and \( f_{ivt}^{OB} \), one obtains

\[
\sum_{v \in V} \sum_{t \in T_v} q_{ivt} \leq \sum_{v \in V} \left( \sum_{t \in T_v^+} f_{iv,t+1}^{OB} + \sum_{j \in N \cup \{d(v)\}} \sum_{t \in T_v} f_{ijv,t+1}^X + \sum_{t \in T \setminus T_v} q_{ivt} \right).
\]

Summing the inventory constraints (10) over \( T \), and taking \( \underline{S}_{it} \) as an under estimator of \( s_{it} \), i.e.,

\( s_{it} \geq \underline{S}_{it} \), it follows from (31) that

\[
s_{ik} + \sum_{v \in V} \left( \sum_{t \in T_v^+} f_{iv,t+1}^{OB} + \sum_{j \in N \cup \{d(v)\}} \sum_{t \in T_v} f_{ijv,t+1}^X + \sum_{t \in T \setminus T_v} q_{ivt} \right) \geq \sum_{t \in T} P_t + \underline{S}_{i,t-1}.
\]

Using the variable upper bound constraints (26) – (30), inequalities (32) imply:

\[
s_{ik} + \sum_{v \in V} \left( \sum_{t \in T_v^+} K_v o_{iv,t+1}^{B} + \sum_{j \in N \cup \{d(v)\}} \sum_{t \in T_v} K_v x_{ijv,t+1} + \sum_{t \in T \setminus T_v} Q_v v_{ivt} \right) \geq \sum_{t \in T} P_t + \underline{S}_{i,t-1}.
\]

Inequality (33) can be viewed as a continuous binary knapsack set \( \{(s, y) \in \mathbb{R}_+^n \times \{0, 1\}^n : \sum_{j=1}^n a_j y_j \leq b + s\} \), see Pochet and Wolsey [21].

Replacing \( s_{ik} \) by its upper bound \( \bar{S}_{ik} \) gives knapsack sets. Valid inequalities for these knapsack sets are valid for \( X^{FCNF} \). Thus for arbitrary \( Q > 0 \), the following Chvátal-Gomory inequalities are valid for \( X^{FCNF} \):

\[
\sum_{v \in V} \left( \sum_{t \in T_v^+} \left[ \frac{K_v}{Q} \right] o_{iv,t+1}^{B} + \sum_{j \in N \cup \{d(v)\}} \sum_{t \in T_v} \left[ \frac{K_v}{Q} \right] x_{ijv,t+1} + \sum_{t \in T \setminus T_v} \left[ \frac{Q_v}{Q} \right] v_{ivt} \right) \geq \left[ \sum_{t \in T} P_t + \underline{S}_{i,t-1} - \bar{S}_{ik} \right].
\]

In Section 5.1 appropriate values for parameter \( Q \) are considered.

**Example 3.1** Inequality (34) is derived for the situation illustrated in Figure 5. A loading port \( i \), a single ship \( v \), and a time interval \( T = [1, 5] \) are given. Taking \( T_v = \{2, 5\} \), implying that the ship can leave the port in time periods 3 or 6, one has by definition \( T_v^+ = \{2, 5\} \) and \( T_v^- = \{2, 5\} \). The ship has capacity \( K_v = 110 \) and loading rate \( Q_v = 60 \). Assume \( S^0_1 = 90 \), \( \bar{S}_{i,5} = 120 \) and \( P_t = 20 \) for all \( t \in T \). Choosing \( Q = Q_v = 60 \), inequality (34) gives

\[
2 o_{i1t3}^B + 2 o_{i1t6}^B + 2 x_{i2t3} + 2 x_{i2t6} + o_{i1t1} + o_{i1t3} + o_{i1t4} \geq \left[ \frac{100 + 90 - 120}{60} \right] = 2
\]

where \( x_{ijvt} = \sum_{j \in N, j \neq i} x_{ijv,t} \).
Two special cases of inequalities (34) lead to simpler inequalities. First, taking $T_v = T$, i.e. that the ship can leave in all time periods, one has in this case $T_v + v = k$ and $T \setminus T_v = \emptyset$. Second, taking $T_v = \emptyset$, implying that the ship cannot leave within the time interval, one has $T_v + v = \emptyset$ and $T \setminus T_v = T$. With $K = \max\{K_v : v \in V\}$ and $Q = \max\{Q_v : v \in V\}$, the corresponding knapsack inequalities are:

$$\sum_{v \in V} \left( \sum_{t \in T} \frac{K_v}{Q} o_{vt} + \sum_{j \in N \cup \{d(v)\}} \sum_{t \in T_v} \frac{K_v}{Q} x_{ijv,t+1} + \sum_{t \in T \setminus T_v} \frac{Q_v}{Q} o_{vt} \right) \geq \left( \sum_{t \in T} P_v + S_{j-1} - S_{ik} \right) / Q,$$

having binary coefficients on all the variables. Inequalities (35) impose a minimum number of ship departures and inequalities (36) impose a minimum number of loading periods at port $i$. These inequalities can also be generalized by aggregating over any nonempty subset of loading ports.

Other inequalities can also be derived for the continuous binary knapsack set. Dividing (33) by $Q > 0$, one obtains:

$$\frac{s_{ik}}{Q} + \sum_{v \in V} \left( \sum_{t \in T_v} \frac{K_v}{Q} o_{vt} + \sum_{j \in N \cup \{d(v)\}} \sum_{t \in T_v} \frac{K_v}{Q} x_{ijv,t+1} + \sum_{t \in T \setminus T_v} \frac{Q_v}{Q} o_{vt} \right) \geq \left( \sum_{t \in T} P_v + S_{j-1} - S_{ik} \right) / Q.$$

Setting $y = \sum_{v \in V} \left( \sum_{t \in T_v} \frac{K_v}{Q} o_{vt} + \sum_{j \in N \cup \{d(v)\}} \sum_{t \in T_v} \frac{K_v}{Q} x_{ijv,t+1} + \sum_{t \in T \setminus T_v} \frac{Q_v}{Q} o_{vt} \right)$, $s = s_{ik}/Q$ and $b = \left( \sum_{t \in T} P_v + S_{j-1} - S_{ik} \right) / Q$, this becomes a basic-mip set of the form: $\{(s, y) \in \}$.
The set \( \mathbb{R}_+ \times \mathbb{Z} : s + y \geq b \) for which the simple mixed integer rounding inequality is derived:

\[
s + fy \geq f \lceil b \rceil \tag{37}
\]

where \( f = b - \lfloor b \rfloor \). For a given \( Q > 0 \), the separation problem (see [17]) for the inequalities (34) and (37) can be solved in polynomial time by finding a minimum capacity cut in a very simple network, where each variable occurring in (34) is represented by an arc, and the arc weight is the value of that variable in the linear relaxation solution.

Discharge ports

The simple mixed integer relaxations used to derive valid inequalities for loading ports, see Section 3.1, are based on ship arcs leaving a subgraph. For discharge ports the network structure is a little more complex since ship arcs entering a subgraph are used. This means that the subgraph includes all three layers of the network, see Figure 6, and the corresponding incident arcs.

Define the time interval \( T = [\ell, k] \subseteq T \) as before. For each ship \( v \), \( T \) is partitioned into three disjoint sets \( R_v^0 \), \( R_v^1 \), and \( R_v^2 \). Also define \( T^2_v = \{ t \in R_v^2 : t - 1 \not\in R_v^2 \} \) and \( T^1_v = \{ t \in R_v^1 \cup R_v^2 : t - 1 \not\in (R_v^1 \cup R_v^2) \} \).

Summing the inventory balance constraints (9) over \( T \) and using \( S_{ik} \) as the lower bound on \( s_{ik} \)
gives the following inequalities written in terms of the partition $R^0_v$, $R^1_v$, and $R^2_v$

$$s_{i,t-1} + \sum_{v \in V} \left( \sum_{t \in R^0_v} q_{v,t} + \sum_{t \in R^1_v} q_{v,t} + \sum_{t \in R^2_v} q_{v,t} \right) \geq \sum_{t \in T} D_{it} + S_{ik}. \quad (38)$$

Summing the flow conservation constraints (22) and (24) over $R^2_v$ and $R^2_v \cup R^1_v$ respectively gives

$$\sum_{t \in R^2_v} f_{i,t}^{OA} = \sum_{t \in R^2_v} \left( \sum_{j \in N \cup \{o(v)\}} \sum_{i \in R^0_v} f_{jiv,t-T_{jiv}}^{X} + f_{jiv,t-1}^{W} - f_{i,t}^{W} \right), \quad (39)$$

$$\sum_{t \in R^2_v \cup R^1_v} \left( f_{i,t}^{OA} + f_{i,t-1}^{OB} - q_{i,t-1} \right) = \sum_{t \in R^2_v \cup R^1_v} \left( f_{i,t}^{OB} + \sum_{j \in N \cup \{d(v)\}} f_{jiv,t}^{X} \right). \quad (40)$$

Simplifying equations (39) by canceling out variables $f_{i,t}^{W}$, and equations (40) by canceling out variables $f_{i,t}^{OB}$, and using the nonnegativity of $f_{i,t}^{W}$, $f_{jiv,t}^{X}$ and $f_{i,t}^{OB}$, we obtain from (38),

$$s_{i,t-1} + \sum_{v \in V} \left( \sum_{t \in R^0_v} q_{v,t} + \sum_{t \in R^1_v} f_{i,t}^{OA} + \sum_{t \in R^2_v} j \sum_{j \in N \cup \{o(v)\}} f_{jiv,t-T_{jiv}}^{X} + \sum_{t \in T} f_{i,t}^{W} + \sum_{t \in T^1_{i}} f_{i,t}^{OB} \right) \geq \sum_{t \in T} D_{it} + S_{ik}. \quad (41)$$

Using the variable upper bound constraints (26) – (30), inequality (41) can be relaxed as follows:

$$s_{i,t-1} + \sum_{v \in V} \left( \sum_{t \in R^0_v} Q_v o_{v,t} + \sum_{t \in R^1_v} K_v o_{v,t}^A + \sum_{t \in R^2_v} j \sum_{j \in N \cup \{o(v)\}} K_v x_{jiv,t-T_{jiv}} + \sum_{t \in T} K_v w_{i,v,t-1} + \sum_{t \in T^1_{i}} K_v o_{v,t}^B \right) \geq \sum_{t \in T} D_{it} + S_{ik}. \quad (42)$$

Constraints (42) have the same structure as constraints (33) and are thus the defining constraints of a continuous binary knapsack set. Setting $s_{i,t-1}$ to its upper bound ($\bar{s}_{i,t-1}$ if $\ell > 1$ and $S^0_v$ if $\ell = 1$) gives an integer knapsack constraint. Using Chvátal-Gomory rounding, we obtain for arbitrary $Q > 0$

$$\sum_{v \in V} \left( \sum_{t \in R^0_v} \left[ \frac{Q_v}{Q} \right] o_{v,t} + \sum_{t \in R^1_v} \left[ \frac{K_v}{Q} \right] o_{v,t}^A + \sum_{t \in R^2_v} j \sum_{j \in N \cup \{o(v)\}} \left[ \frac{K_v}{Q} \right] x_{jiv,t-T_{jiv}} + \sum_{t \in T} \left[ \frac{K_v}{Q} \right] w_{i,v,t-1} + \sum_{t \in T^1_{i}} \left[ \frac{K_v}{Q} \right] o_{v,t}^B \right) \geq \frac{\sum_{t \in T} D_{it} - \bar{s}_{i,t-1} + S_{ik}}{Q}. \quad (43)$$

Three special cases of these inequalities are obtained by setting $R^2_v = T$, $R^1_v = T$, and $R^0_v = T$ respectively. Choosing $\bar{K} = \max \{ K_v : v \in V \}$ and $\bar{Q} = \max \{ Q_v : v \in V \}$, one obtains:

$$\sum_{v \in V} \left( \sum_{j \in N \cup \{o(v)\}} \sum_{t \in T} x_{jiv,t-T_{jiv}} + w_{i,v,t-1} + o_{v,t}^B \right) \geq \frac{\sum_{t \in T} D_{it} - \bar{s}_{i,t-1} + S_{ik}}{\bar{K}}, \quad (44)$$

$$\sum_{v \in V} \left( \sum_{t \in T} o_{v,t}^A + o_{v,t}^B \right) \geq \frac{\sum_{t \in T} D_{it} - \bar{s}_{i,t-1} + S_{ik}}{\bar{K}}, \quad (45)$$

$$\sum_{v \in V} \sum_{t \in T} o_{v,t} \geq \frac{\sum_{t \in T} D_{it} - \bar{s}_{i,t-1} + S_{ik}}{\bar{Q}}. \quad (46)$$
Inequalities (44) establish the minimum number of arrivals at port \(i\), (45) establish the minimum number of times a ship must start operating, and inequalities (46) impose a minimum number of operations. Inequalities (44) and (46) can be generalized for any nonempty subset of discharge ports.

**Example 3.2** Inequality (43) is derived for the situation illustrated in Figure 6 based on the entering arcs crossing the cut-set shown. A discharge port \(i\), a single ship \(v\), and a time interval \(T = [1, 5]\) are given. \(T\) is partitioned into \(R^0_v = \{3\}\), \(R^1_v = \{4\}\), and \(R^2_v = \{1, 2, 5\}\). The ship has capacity \(K_v = 110\) and its discharge rate is \(Q_v = 60\). Assume \(S^0_i = 40\), \(S^5_i = 10\) and \(D_{it} = 20, \forall t \in T\). Choosing \(Q = Q_v = 60\) then gives

\[
o_{iv1} + o_{iv2} + 2o_{iv4} + o_{iv5} + 2\pi_{iv3} + 2w_{iv2} + 2o_{iv3} \geq \left\lceil \frac{100 - 40 + 10}{60} \right\rceil = 2
\]

where \(\pi_{ivt} = \sum_{j \in N \cup o(v), j \neq i; u - T_{ji} = t} x_{jivu}\).

## 4 Lot-sizing relaxations

Single item lot-sizing is concerned with the production of a single product for which there is a demand \(D_t\) in each time period. To model the problem as a mixed integer program, one defines the production \(q_t\) in each period, the stock of the product \(s_t\) at the end of the period and a 0-1 set-up variable taking the value \(o_t = 1\) if there is production in the period \((q_t > 0)\). Additional aspects involve upper and lower bounds on the stock, production capacity implying an upper bound \(Q_t\) on the amount produced per period and start-up variables \(o^A_t = 1\) if period \(t\) is the first period of an interval of set-ups \((o_t = 1\) and \(o_{t-1} = 0)\). One can also view \(o_t\) as an an integer variable, in which case it represents the number of batches of maximum size \(Q\) that are required to produce \(q_t\). As we will show below, this corresponds very closely to the situation in a discharge port.

### 4.1 Constant capacitated lot-sizing relaxations

The first lot-sizing relaxations that we derive correspond to one level of the fixed charge network at discharge port \(i\), see Figure 7 by imposing a constant upper bound on the \(q_{ivt}\) variables.?? The
constraints (7) – (9), (11), (12), (17) lead to the relaxation

\[ s_{i,t-1} + \sum_{v \in V} q_{ivt} = D_{it} + s_{it}, \quad \forall t \in T, \tag{47} \]

\[ 0 \leq q_{ivt} \leq Q_v o_{ivt}, \quad \forall v \in V, t \in T, \tag{48} \]

\[ \sum_{v \in V} o_{ivt} \leq B_{it}, \quad \forall t \in T, \tag{49} \]

\[ s_{it} \leq s_{it} \leq \bar{s}_{it}, \quad \forall t \in T, \tag{50} \]

\[ s_{i0} = S^0_i, \tag{51} \]

\[ o_{ivt} \in \{0, 1\}, \quad \forall v \in V, t \in T. \tag{52} \]

For the constant capacity lot-sizing set, the upper bounds on the inventory level variables \( s \) are relaxed. The lower bounds on the same variables can then be eliminated. To do so, one first updates the bounds as follows:

\[ \bar{s}^M_{it} = S^0_i \text{ if } t = 0, \text{ and } \bar{s}^M_{it} = \max \{ \bar{s}^M_{it}, \bar{s}_{i,t-1} - D_{it} \} \text{ if } t \in T \]

where \( \bar{s}^M_{it} \) is the modified lower bound and \( s_{i0} = S^0_i \). Now, a new net inventory level variable \( \tilde{s}_{it} \) and demand \( \tilde{D}_{it} \) can be defined as:

\[ \tilde{s}_{it} = 0 \text{ if } t = 0, \text{ and } \tilde{s}_{it} = s_{it} - \bar{s}^M_{it} \text{ if } t \in T, \text{ and } \tilde{D}_{it} = D_{it} - \bar{s}^M_{i,t-1} + \bar{s}^M_{it}. \tag{54} \]

Based on the subset (47) – (52) and setting \( \tilde{q}_{it} = \sum_{v \in V} q_{ivt}, \tilde{o}_{it} = \sum_{v \in V} o_{ivt}, \text{ and } \bar{Q} = \max \{Q_v : v \in V \} \), one obtains the constant capacitated lot-sizing set, LSCC, for discharge port \( i \):

\[ \tilde{s}_{i,t-1} + \tilde{q}_{it} = \tilde{D}_{it} + \tilde{s}_{it}, \quad \forall t \in T, \tag{55} \]

\[ \tilde{q}_{it} \leq \bar{Q} \tilde{o}_{it}, \quad \forall t \in T, \tag{56} \]

\[ \tilde{q}_{it}, \tilde{s}_{it} \geq 0, \quad \forall t \in T, \tag{57} \]

\[ \tilde{o}_{it} \in Z^+_1 \quad \forall t \in T. \tag{58} \]
If it is assumed that the berth capacity $B_{it} = 1$, then (58) becomes $\tilde{o}_{it} \in \{0, 1\}$.

For discharge port $i$, a relaxation of (55)-(58), known as the Wagner-Whitin relaxation $\text{WWCC}$, can now be given:

$$\tilde{s}_{i,k-1} + \sum_{u=k}^{t} \tilde{o}_{iu} \geq \sum_{u=k}^{t} \tilde{D}_{iu},$$

$$\forall k \in T, t \in T, k \leq t,$$

$$\tilde{s}_{it} \geq 0, \tilde{o}_{it} \in Z^+_1,$$

$$\forall t \in T.$$

A complete polyhedral description of the convex hull of solutions of $\text{WWCC}$ is known, as well as a polynomial size extended formulation, as well as valid inequalities for $\text{LSCC}$, see [19, 20]. For a comprehensive survey on the valid inequalities for these sets, see Pochet and Wolsey [21]. Valid inequalities for $\text{LSCC}$ and $\text{WWCC}$ can be converted back into valid inequalities for $X_{\text{FCNF}}$ using the linear transformations:

$$\tilde{s}_{it} = s_{it} - \tilde{S}_{it}^M,$$

$$\tilde{o}_{it} = \sum_{v \in V} o_{ivt},$$

$$\tilde{q}_{it} = \sum_{v \in V} q_{ivt},$$

$$\tilde{D}_{it} = \sum_{v \in V} D_{ivt}.$$

**Example 4.1** Consider an instance based on Figure 7 over five time periods $T = \{1, 5\}$ with demands $D_i = (3, 2, 3, 4, 2)$, lower bounds on the inventory levels $S_i = (2, 2, 2, 2, 2)$, initial inventory $S_{i0} = 6$ and the capacity of the largest ship $Q = 5$.

Calculating the modified lower bounds on the inventory levels according to (53) and the demand according to (54) gives $\tilde{S}_{it}^M = (6, 3, 2, 2, 2)$ and $\tilde{D}_{it} = (0, 1, 3, 4, 2)$. For the corresponding $\text{WWCC}$ relaxation, a valid inequality is:

$$\tilde{s}_{i2} \geq 3(1 - \tilde{o}_{i3}) + 1(2 - \tilde{o}_{i3} - \tilde{o}_{i4} - \tilde{o}_{i5}).$$

Transforming back to the original variables $\tilde{s}_{i2} = s_{i2} - \tilde{S}_{i2}^M$, $\tilde{o}_{it} = \sum_{v \in V} o_{ivt}$ and collecting the terms one obtains the valid inequality for $X_{\text{FCNF}}$:

$$s_{i2} \geq 7 - 4 \sum_{v \in V} o_{iv3} - \sum_{v \in V} o_{iv4} - \sum_{v \in V} o_{iv5}.$$
To formulate this problem as a lot-sizing problem, new variables $\hat{s}_{it} = S_i - s_{it}$ are introduced that measure the spare stock capacity available at period $t$ in port $i$, where $S_i = \max\{S_i^0, \max_{t \in T} \bar{S}_t\}$.

This leads to the following equivalent formulation

$$
\hat{s}_{i,t-1} + \sum_{v \in V} q_{ivet} = P_{it} + \hat{s}_{it}, \quad \forall t \in T,
$$

$$
0 \leq q_{ivet} \leq Q_v o_{ivet} \quad \forall v \in V, t \in T,
$$

$$
\sum_{v \in V} o_{ivet} \leq B_{it} \quad \forall t \in T,
$$

$$
\bar{S}_i - \bar{s}_{it} \geq \hat{s}_{it} \geq \bar{S}_i - \bar{s}_{it} \quad \forall t \in T,
$$

$$
\hat{s}_{i0} = \bar{S}_i - s_{i0}^0,
$$

$$
o_{ivet} \in \{0, 1\}, \quad \forall v \in V, t \in T.
$$

This formulation can now be used to derive the same relaxations as for the discharge ports.

4.2 Two level lot-sizing relaxations

The two level relaxations are derived from two levels of the fixed charge network, see Figure 8. In multi-level lot-sizing problems it is useful to consider the concept of echelon stock. The echelon stock of an item at a period $t$ is the total stock of that item within the system at period $t$. For a discharge port $i$ such as the one depicted in Figure 8, the echelon stock at the end of period $t$ is given by the stock held in port $i$, $s_{it}$, plus the stock held in the ships operating at port $i$ during time periods $t$ and $t + 1$.

Extending the lot-sizing relaxations defined in Section 4.1, the two level lot-sizing set (2LLS)
for discharge port $i$ can be defined as

\[
\tilde{f}_{i,t}^{OA} + \tilde{f}_{i,t-1}^{OB} = \tilde{q}_{i,t-1} + \tilde{f}_{i,t}^{X}, \quad \forall t \in T, \quad (65)
\]

\[
\tilde{f}_{i,t}^{OA} \leq K \tilde{o}_{i,t}^{A}, \quad \forall t \in T, \quad (66)
\]

\[
\tilde{f}_{i,t}^{OA}, \tilde{f}_{i,t}^{OB}, \tilde{f}_{i,t}^{X} \geq 0, \quad \forall t \in T, \quad (67)
\]

\[
\tilde{o}_{i,t}^{A} \in \mathbb{Z}_{+}, \quad \forall t \in T, \quad (68)
\]

and (55) – (58) (69)

where

\[
\tilde{f}_{i,t}^{OA} = \sum_{v \in V} f_{i,t}^{OA}, \quad \tilde{f}_{i,t}^{OB} = \sum_{v \in V} f_{i,t}^{OB}, \quad \tilde{f}_{i,t}^{X} = \sum_{v \in V} \sum_{j \in N \cup \{d(v)\}} f_{i,t}^{X}, \quad \tilde{q}_{i,t} = \sum_{v \in V} q_{i,t}, \quad \tilde{o}_{i,t}^{A} = \sum_{v \in V} o_{i,t}^{A}
\]

and $K = \max\{K_v : v \in V\}$. Constraints (65) are the flow balance constraints (24) summed over $v$.

Summing together the constraints (65) and (55) and introducing the echelon stock variable $e_{i,t} = \tilde{f}_{i,t+1}^{OB} + \tilde{s}_{i,t}$, one obtains the relaxation:

\[
e_{i,t-1} + f_{i,t}^{OA} = D_{i,t} + e_{i,t} + f_{i,t+1}^{X}, \quad \forall t \in T,
\]

\[
\tilde{f}_{i,t}^{OA} \leq K \tilde{o}_{i,t}^{A}, \quad \forall t \in T,
\]

\[
e_{i,t}, \tilde{f}_{i,t}^{OA} \geq 0, \tilde{o}_{i,t}^{A} \in \mathbb{Z}_{+}, \quad \forall t \in T.
\]

From this we again obtain a Wagner-Whitin constant capacity relaxation:

\[
e_{i,k-1} + K \sum_{u=k}^{t} \tilde{o}_{i,u}^{A} \geq \sum_{u=k}^{t} D_{i,u}, \quad \forall k \in T, t \in T, k \leq t,
\]

\[
e_{i,t} \geq 0, \tilde{o}_{i,t}^{A} \in \{0, 1\}, \quad \forall t \in T,
\]

Again valid inequalities for this relaxation, denoted 2LWWCC, can be derived, and then converted into valid inequalities for $X^{FCNF}$.

In order to derive similar inequalities for loading port $i$, new variables, $f_{i,t}^{OA}, f_{i,t}^{OB}$, and $f_{ij,t}$ are defined. These variables indicate the unused capacity of each ship operating at that port, i.e.

\[
f_{i,t}^{OA} = K_v o_{i,t}^{A} - f_{i,t}^{OA}, \quad f_{i,t}^{OB} = K_v o_{i,t}^{B} - f_{i,t}^{OB}, \quad f_{ij,t}^{X} = K_v x_{ij,t} - f_{ij,t}^{X}.
\]

Thus two level lot-sizing relaxations can be derived at loading ports.

### 4.3 Lot-sizing with start-up relaxations

When deriving lot-sizing relaxations from the standard formulation, as in Section 4.1, it is not possible to handle start-ups since the variable $o_{i,t}$ does not give information on whether ship $v$
operated at port $i$ at time period $t - 1$ or not. In the fixed charge network flow problem, $o_{ivt}^A$ can be interpreted as a start-up variable and can be used to derive valid inequalities.

In the 0-1 case, a start-up variable $o_{ivt}^A = 1$ if $o_{ivt} = 1$ and $o_{iv,t-1} = 0$, while in the integer case $o_{ivt}^A$ is the increase of $o_{ivt}$ from period $t - 1$ to $t$.

This can be expressed by

$$o_{ivt}^A \leq o_{ivt}, \quad \forall t \in T,$$

(70)

$$o_{ivt}^A \geq o_{ivt} - o_{iv,t-1}, \quad \forall t \in T,$$

(71)

$$o_{ivt}, o_{ivt}^A \in \{0, 1\}, \quad \forall t \in T.$$  

(72)

Constraints (70) ensure that ship $v$ starts operating if there is a start-up, while constraints (71) force a start-up if the ship operates in the current time period and did not operate in the previous time period.

Several lot-sizing relaxations with start-ups can be derived by adding (70) – (72) to an existing lot-sizing set. In particular, valid inequalities can be derived for the capacitated lot-sizing set with start-ups, see Constantino [9], and then used to derive valid inequalities for $X^{FCNF}$.

Here we derive a discrete constant capacity lot-sizing with start-ups relaxation (DLSCCS), for which valid inequalities have been proposed in [31].

Constraints (70) – (72) are aggregated by summing over $v$. This gives

$$\delta_{it}^A \leq \delta_{it}, \quad \forall t \in T,$$

(73)

$$\delta_{it}^A \geq \delta_{it} - \delta_{i,t-1}, \quad \forall t \in T,$$

(74)

$$\delta_{it}, \delta_{it}^A \in \{0, 1, \ldots, B_{it}\}, \quad \forall t \in T,$$

(75)

where $\delta_{it}^A = \sum_{v \in V} o_{ivt}^A$ and $\delta_{it} = \sum_{v \in V} o_{ivt}$. (If it is assumed that the berth capacity $B_{it} = 1$, the aggregated variables are still binary).

Now let $\tilde{O}_{it} = \left\lceil \frac{\sum_{u=1}^{t} \hat{D}_{iu}}{Q} \right\rceil$, where $\hat{D}_{iu}$ is the modified demand from (54) and $Q = \max\{Q_v : v \in V\}$ is the largest ship capacity. Also set $\delta_{it} = \tilde{O}_{it} - \tilde{O}_{i,t-1}$. Note that $\tilde{O}_{it}$ is a lower bound on the number of operating periods needed during the first $t$ periods. The set DLSCCS is obtained by adding the constraints

$$\sum_{u=1}^{t} \delta_{iu} \geq \tilde{O}_{it}, \quad \forall t \in T,$$

(76)

to constraints (73) – (75).

The following set of inequalities was proved to be valid for DLSCCS by van Eijl and van Hoesel [31] in the case where $\delta_{it}$ and $B_{it}$ are 0-1, and thus $\hat{\delta}_{it}$ and $\delta_{it}^A$ are 0-1 variables.

Proposition 4.2 Consider a time interval $[k, \ell] \subseteq T$ with $\delta_{it} = 1$. Let $\sum_{t=k}^{\ell} \delta_{it} = p > 0$ and let
$t_1 < t_2 < \cdots < t_p = \ell$ be the periods in $[k, \ell]$ in which $\delta_{it} = 1$. The inequality

$$\sigma_{i,k-1} + \sum_{j=1}^{p} (\tilde{o}_{i,k+j-1} + \tilde{o}^A_{i,k+j} + \cdots + \tilde{o}^A_{p}) \geq p$$

is valid for DLSCCS, where $\sigma_{it} = \sum_{u=1}^{t} \tilde{o}_{iu} - \tilde{O}_{it} \geq 0$ and $\sigma_{i0} = 0$.

**Example 4.3** Consider the data from Example 4.1. Since $\tilde{D}_{it} = (0,1,3,4,2)$ and $\tilde{Q} = 5$ it follows that $\delta_{it} = (0,1,0,1,0)$. Let $[k, \ell] = [1,4]$. This gives $t_1 = 2$, $t_2 = 4$ and $p = \sum_{t=k}^{\ell} \tilde{O}_{it} = 2$. A valid inequality derived from (77) is then

$$\sigma_{i0} + (\tilde{o}_{i1} + \tilde{o}^A_{i1}) + (\tilde{o}_{i2} + \tilde{o}^A_{i2} + \tilde{o}^A_{i3}) \geq 2 \Rightarrow \tilde{o}_{i1} + \tilde{o}_{i2} + \tilde{o}^A_{i2} + \tilde{o}^A_{i3} + \tilde{o}^A_{i4} \geq 2$$

Hence the following inequality is valid for $X^{FCNF}$:

$$\sum_{v \in V} o_{v1} + \sum_{v \in V} o_{v2} + \sum_{v \in V} o^A_{v2} + \sum_{v \in V} o^A_{v3} + \sum_{v \in V} o^A_{v4} \geq 2.$$  

## 5 Computational results

This section presents some of the computational experiments carried out to test different strategies for the solution of instances of the maritime inventory routing problem. The strategies tested include the comparison of the two mathematical formulations presented in Section 2, the effectiveness of the inclusion of those valid inequalities discussed in Sections 3 and 4 and the use of branching priorities.

First the standard and FCNF formulations with and without the inclusion of valid inequalities are compared. This initial study leads to the selection of some relevant inequalities. Then, taking the two formulations tightened with the selected inequalities, different branching priorities are tested. Thus for each formulation, several different combinations of valid inequalities and branching priorities are tested. Finally, the scalability of the approaches are tested by changing the discretization of the time periods and the length of the time horizon.

The instances used were generated from seven instances based on real data. The number of ports and ships of each instance is given in the second column of Table 1. The time horizon is 30 days. Traveling, operating and waiting costs are time invariant.

All tests were run on a computer with processor Intel Core 2 Duo, CPU 2.2GHz, with 4GB of RAM using the optimization software Xpress Optimizer Version 21.01.00 with Xpress Mosel Version 3.2.0. Unless stated otherwise, all the inequalities used to tighten the formulations were added a priori to the MIP model which was then fed to the MIP solver.

In the last six columns of Table 1 we provide summary information for the two formulations considered. Columns “Rows” and “Columns” indicate the total number of constraints and variables,
respectively. The column “Int. Var.” indicates the number of integer variables. In parentheses we provide the corresponding value after the preprocessing phase.

<table>
<thead>
<tr>
<th>Inst.</th>
<th>Standard Formulation</th>
<th>FCNF Formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Rows</td>
<td>Columns</td>
</tr>
<tr>
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<td>(662)</td>
</tr>
<tr>
<td>B</td>
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<td>1128</td>
</tr>
<tr>
<td></td>
<td>(757)</td>
<td>(795)</td>
</tr>
<tr>
<td>C</td>
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<td>1756</td>
</tr>
<tr>
<td></td>
<td>(1197)</td>
<td>(1542)</td>
</tr>
<tr>
<td>D</td>
<td>2138</td>
<td>2355</td>
</tr>
<tr>
<td></td>
<td>(1445)</td>
<td>(1863)</td>
</tr>
<tr>
<td>E</td>
<td>2138</td>
<td>2367</td>
</tr>
<tr>
<td></td>
<td>(1446)</td>
<td>(1878)</td>
</tr>
<tr>
<td>F</td>
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<td>2548</td>
</tr>
<tr>
<td></td>
<td>(1696)</td>
<td>(2237)</td>
</tr>
<tr>
<td>G</td>
<td>5836</td>
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</tr>
<tr>
<td></td>
<td>(3150)</td>
<td>(4165)</td>
</tr>
</tbody>
</table>

Table 1: Summary statistics for the base instances using the two models (with and without preprocessing).

5.1 Formulations, valid inequalities and reformulations

The following valid inequalities and reformulations have been tested:

**Knapsack inequalities.** These inequalities refer to (34) for the loading ports and (43) for discharge ports. In both cases $T$ includes either the first period or the last period, that is, $T = 1, \ldots , t$ or $T = t, \ldots , |T|$, $t \in T$. Inequalities (34) are generated for case $T_v = T$, for all $Q \in \bigcup_{v \in V} \{K_v\}$, and for case $T_v = \emptyset$, for all $Q \in \bigcup_{v \in V} \{Q_v\}$. Inequalities (43) are generated for the cases $R^2_v = T$, $R^1_v = T$, and $R^0_v = T$ for $Q \in \bigcup_{v \in V} \{K_v\}$ in the first two cases and for $Q \in \bigcup_{v \in V} \{Q_v\}$ in the former one. These inequalities will henceforth be denoted $K$.

**Mixed integer rounding inequalities.** These inequalities are stated in (37) and are generated for all $Q \in \bigcup_{v \in V} \{Q_v\}$. They are added as cuts, (valid inequalities that cut off the current fractional solution), and will henceforth be denoted $M$.

**Wagner-Whitin constant capacitated lot-sizing reformulations.** These reformulations are given in Pochet and Wolsey [21] (denoted by XFormWWCC for the constant capacitated case and XFormWWU for the uncapacitated case) for the WWCC relaxation described in Section 4.1 and the two-level relaxation 2LWWCC described in Section 4.2. These reformulations are denoted $W$.

**Inequalities for lot-sizing with start-up relaxations.** These inequalities are stated in (77) and will henceforth be denoted $D$. These inequalities consider every subset $[k, \ell]$ of $T$.

Table 2 gives some characteristics of each instance and provides information on the lower bounds.
obtained with the standard formulation. The first column specifies the instance, and the second column contains the optimal value. The last four columns present the lower bounds obtained with the linear relaxation, denoted \( L \), of the standard formulation with the inclusion of additional valid inequalities. \( L + K \) means that the valid inequalities \( K \) are added to the formulation, \( L + K, W \) means that valid inequalities \( K \) and \( W \) are added.

<table>
<thead>
<tr>
<th>Inst.</th>
<th>Opt.</th>
<th>( L )</th>
<th>( X )</th>
<th>( K )</th>
<th>( W )</th>
<th>( K, W )</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
<td>B</td>
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<td>21.8</td>
<td>78.6</td>
<td>48.4</td>
<td>91.4</td>
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<td>16.0</td>
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<td>46.3</td>
<td>85.1</td>
</tr>
<tr>
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<td>25.8</td>
<td>74.5</td>
<td>43.9</td>
<td>85.4</td>
</tr>
<tr>
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<td>105.2</td>
<td>11.3</td>
<td>81.3</td>
<td>23.5</td>
<td>79.2</td>
</tr>
<tr>
<td>G</td>
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<td>19.1</td>
<td>92.0</td>
<td>43.0</td>
<td>71.6</td>
</tr>
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</table>

Table 2: Lower bounds based on the standard formulation.

Table 3 shows the results obtained with some of the more interesting and/or effective combinations of valid inequalities and reformulations for the FCNF formulation. To ease the presentation, \( \Omega \) is introduced to denote the inclusion of all valid inequalities, i.e. \( \Omega = K, M, W, D \) and \( \Omega - \Delta \) denotes the inclusion of all valid inequalities except \( \Delta \), where \( \Delta \in \{ K, M, W, D \} \).

<table>
<thead>
<tr>
<th>Inst.</th>
<th>Opt.</th>
<th>( L )</th>
<th>( X )</th>
<th>( W )</th>
<th>( D )</th>
<th>( K, M )</th>
<th>( \Omega - K, M )</th>
<th>( \Omega - W )</th>
<th>( \Omega - D )</th>
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<tr>
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<td>53.0</td>
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<td>100</td>
</tr>
<tr>
<td>B</td>
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<td>263.4</td>
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</tr>
<tr>
<td>D</td>
<td>357.9</td>
<td>204.1</td>
<td>52.6</td>
<td>52.9</td>
<td>10.9</td>
<td>58.8</td>
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<td>61.8</td>
<td>61.3</td>
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<td>57.6</td>
<td>58.5</td>
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<td>66.1</td>
<td>79.6</td>
<td>79.7</td>
<td>80.1</td>
</tr>
</tbody>
</table>

Table 3: Lower bounds based on the FCNF formulation.

As expected, the FCNF formulation provides better bounds. It can also be observed that best bounds when only one type of inequalities is tested were obtained with the inclusion of \( K \) inequalities and \( M \) cuts. \( K \) and \( M \) are considered in the same type of inequality since \( K \) can be
generated from $M$. On the other hand, dropping reformulations $W$ or dropping inequalities $D$ leads to a small deterioration in bound. This suggests that a good formulation should be based on some inequalities $K$ and $M$. However, extended testing (not reported in Table 3) showed that it is necessary to add many $M$ cuts to get significant improvements on the lower bounds. That experience also showed that most of the gap closed by $K$ and $M$ can be closed by $K$.

Table 4 gives the average integrality gap over the seven instances, where $\text{gap} = \frac{\text{Opt} - \text{LB}}{\text{Opt}} \times 100$ and $\text{LB}$ is the value of the lower bound provided by the corresponding relaxation. The $X$ indicates the use of Xpress cuts. When $X$ is present, the gap reported is the gap at the root node after the inclusion of cuts from Xpress. For example, $L + \Omega, X$ under FCNF formulation means that the gap is measured at the root node when the FCNF formulation is used with the addition of all valid inequalities (or reformulations) and Xpress cuts are added.

<table>
<thead>
<tr>
<th></th>
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<th>FCNF formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
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<tr>
<td>$L + X$</td>
<td>57.5</td>
<td>8.8</td>
</tr>
<tr>
<td>$L + \Omega$</td>
<td>14.4</td>
<td>11.4</td>
</tr>
<tr>
<td>$L + \Omega, X$</td>
<td>10.1</td>
<td>6.4</td>
</tr>
</tbody>
</table>

Table 4: Average integrality gaps for both formulations.

The valid inequalities added to the standard formulation are much stronger than the general cuts added by Xpress, while the general cuts by Xpress gives a stronger FCNF formulation. Combining valid inequalities and cuts from Xpress further reduces the gap of both formulations.

### 5.2 Branching strategies

It is well known that branching decisions within a Branch and Bound algorithm may have great influence on the performance of the algorithm. Usually, solvers, as Xpress Optimiser, allow the user to define his own branching scheme. One possible branching strategy is to establish different branching priorities on variables. Here we followed this approach by considering new variables (resulting from aggregation of the original variables) providing information related to the total number of visits each ship does to each port.

Based on the results in Table 2 and 3 and related runs it was decided to use the following strategies for further tests:

- $Sx$ - set highest priority to variables $Sx_{it} = \sum_{t \in T} \sum_{j \in N \cup \{o(v)\}} x_{jivt}$ that represent the number of times ship $v$ visits port $i$;

- $So^A$ - set highest priority to variables $So^A_{it} = \sum_{t \in T} o^A_{it}$ that represent the number of start-ups of ship $v$ at port $i$. 

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For the standard formulation only strategy $S_x$ can be used. We tested the use of this strategy combined with the inclusion of inequalities $K$. For the FCNF formulation both strategies have been tested. These strategies were combined with the inclusion of inequalities $K$ and $D$. The choice of $D$ inequalities was motivated by the possibility of combining valid inequalities involving the start-up variables $o^A$ with the branching strategy based on the same set of variables. Inequalities $K$ are included a priori in the formulation while inequalities $D$ are added to the cut pool. Since slightly better results were obtained with $So^A$ for the harder instances, only results for $So^A$ are provided. Tables 5 and 6 show the results for the standard and FCNF formulations, respectively.

Each pair $(V, B)$ in the header row of the tables indicates the combination of valid inequalities $(V)$ and branching priority $(B)$ used. The symbol – means that no inequality or branching priority is added. For each such pair, the time $T$ in seconds and the number of branch and bound nodes $N$ is given. A * means that the optimal solution could not be found within a three hours limit.

<table>
<thead>
<tr>
<th>Inst.</th>
<th>$T$</th>
<th>$N$</th>
<th>$T$</th>
<th>$N$</th>
<th>$T$</th>
<th>$N$</th>
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<tr>
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<td>*</td>
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<td>24278</td>
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Table 5: Branching priorities for branch and bound with the standard formulation.

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<th>$T$</th>
<th>$N$</th>
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<td>11</td>
<td>7</td>
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<td>711</td>
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<td>*</td>
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<td>297</td>
<td>111</td>
<td>119</td>
<td>188</td>
<td>393</td>
</tr>
</tbody>
</table>

Table 6: Branching priorities for branch and bound with the FCNF formulation.
Tables 5 and 6 show that the use of branching priorities is essential to solve the instances tested. An efficient approach is the use of the FCNF formulation with the combination of inequalities \( K \) and \( D \) with a branching priority on the \( \sigma^A \) variables.

In order to further test this strategy more computational experiments were conducted. Five new instances for each base instance were created by randomly generating the initial inventory, using a uniform distribution on \([\sum_{i=1}^{m}, S_i]\) in each port \( i \in N \). We choose the first five feasible instances generated. The average solution times and average number of nodes over the five instances are given in Table 7. The random instances based on initial instance G turned out to be much harder than G. Only two of them were solved within the limit of three hours, so these instances are not presented in the table. The ** in column \((K, -)\) indicates that some of the corresponding five instances were not solved within three hours.

<table>
<thead>
<tr>
<th>Inst.</th>
<th>Standard formulation</th>
<th>FCNF formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>((K, Sx))</td>
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<tr>
<td>C</td>
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<tr>
<td>F</td>
<td>**</td>
<td>**</td>
</tr>
</tbody>
</table>

Table 7: Branching priorities and \( D \) inequalities for random instances.

Using the FCNF formulation with inequalities \( K \) and \( D \) (using \( \sigma^A \) variables), and using branching priority \( So^A \) performs well on most instances. The good performance of this approach based on the “start-ups” is also reinforced with the results given in Section 5.3.

### 5.3 Scalability study

Seven larger instances were constructed to test the time discretization. Each day is split into two periods, doubling the number of periods. The demand/production of the first new period is set to zero and the demand of the second is set to the demand/production of the day. The same settings as in Table 7 have been used. Table 8 gives the results for these instances. Again it can be seen that the use of branching priorities is essential, and the best results are obtained when inequalities \( K \) and \( D \) are added. The FCNF formulation with the addition of inequalities \( K \) and \( D \), and with the use of the branching priority \( So^A \) is particularly successful for large test instances.
Finally, different time horizons were tested. In order to extend the time horizon it was necessary to change the port consumption rates $D_{it}$ and production rates $P_{it}$ for the instances. The results using the FCNF formulation with inequalities $K$ added a priori and inequalities $D$ added to the cut pool, and the branching priority $S/o^A$ are given in Table 9. A * means that the optimal solution could not be found within a three hours limit. For the case of instance $G$ with 45 days, the integrality gap after three hours is about 25%, and for 60 days no feasible solution was found within the running time limit.

<table>
<thead>
<tr>
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<td>*</td>
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<td>*</td>
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<td>*</td>
<td>*</td>
<td>*</td>
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<td>7410</td>
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Table 8: Results for the two periods per day case.

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<tr>
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</table>

Table 9: Results for the 30, 45 and 60 days using the FCNF formulation.
6 Concluding remarks

A maritime inventory routing problem with varying production and consumption rates is studied in this paper. Two discrete time formulations are introduced, a standard formulation and a fixed charge network flow formulation (FCNF) that models the ship sequence of actions as a path on a given network. These formulations are strengthened using valid inequalities from (mixed) integer sets that arise as relaxations of these two formulations. In particular, several lot-sizing relaxations are derived for the FCNF formulation. It has been observed in studying lot-sizing problems that valid inequalities linking stocks and 0-1 set-up variables indicating whether a period is a production period can often be strengthened by using additional 0-1 variables indicating a start-up period at the beginning of one or more production periods. Taking production periods to correspond to loading/discharge periods and ship arrivals to correspond to the start-up variables mean that such strengthening is also possible here. In addition a branching strategy based on these start-up variables turns out to be better than a similar strategy based on the set-up variables.

With the FCNF formulation tightened with valid inequalities and using a branching strategy based on the start-up variables instances with up to 60 periods could be solved to optimality.

As future research, it would be interesting to investigate heuristics that could provide feasible solutions quickly, since there are instances with few feasible solutions for which it is hard to get good upper bounds early in the search. Combining such heuristics with a branch and cut approach might be fruitful. Another direction is to investigate further valid inequalities for different lot-sizing models as well as valid inequalities for the ship routing aspect of the problem. Investigation of the possibility of using the valid inequalities presented here together with column generation in a branch and price and cut framework is another interesting path for further research.

References


